# Effective Approximation based on Boundary Measurements

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## An Inverse Multiscale Problem

**Objectives**

We aim at developping a **methodology to contruct coarse approximations** of highly oscillating PDEs when coefficients are *not* known and only *boundary (possibly agregated) measurements* of itssolutions are available. This methodology is **inspired by homogenization theory** but overcome some of its limitations (e.g. periodicity assumptions) and is more **versatile** (e.g. valid outside the homogenization regime).

Assume we are able to measure observables $\mathcal{O}(A_\varepsilon, g)$ associated to the multiscale Schrödinger equation (1) for a few selected loadings $g_1, ..., g_P$.

$$\begin{cases} -\text{div}\,(A_\varepsilon \nabla u_\varepsilon) = 0 \text{ in } \mathcal{D}, \\ (A_\varepsilon \nabla u_\varepsilon) \cdot n = g \text{ on } \partial\mathcal{D}. \end{cases} \tag{1}$$

How can we define an *effective constant coefficient* $\overline{A}$ such that solutions $u_\varepsilon$ to (1) for new RHS are *well* approximated by the solutions $\overline{u}$ to the coarse diffusion problem

$$\begin{cases} -\text{div}\,(\overline{A} \nabla \overline{u}) = 0 \text{ in } \mathcal{D}, \\ (\overline{A} \nabla \overline{u}) \cdot n = g \text{ on } \partial\mathcal{D}. \end{cases} \tag{2}$$

In this study, the observable is the compliance defined by $\mathcal{O}(A_\varepsilon, g) = \frac{1}{2}\int_{\partial\mathcal{D}} g u_\varepsilon(g)$. Another choice could be boundary measurements $u_\varepsilon(g)|_{\partial\mathcal{D}}$.

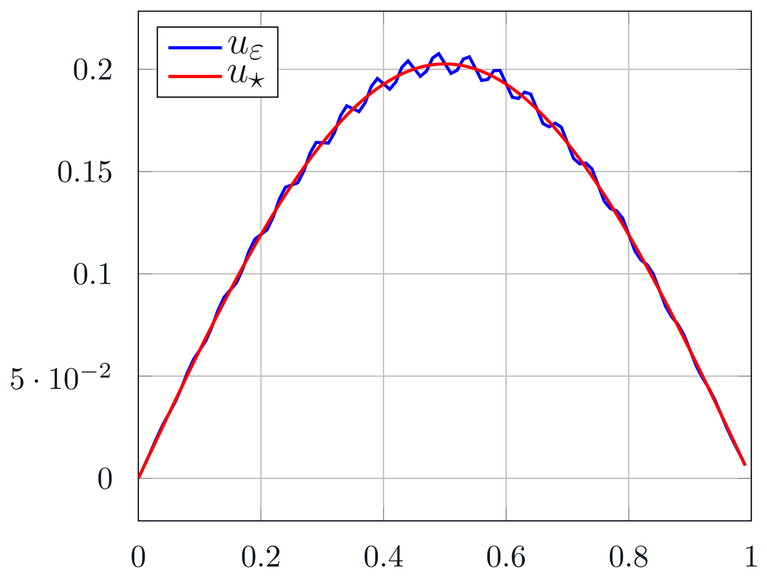


**Fig 1.** : Diffusion Solutions

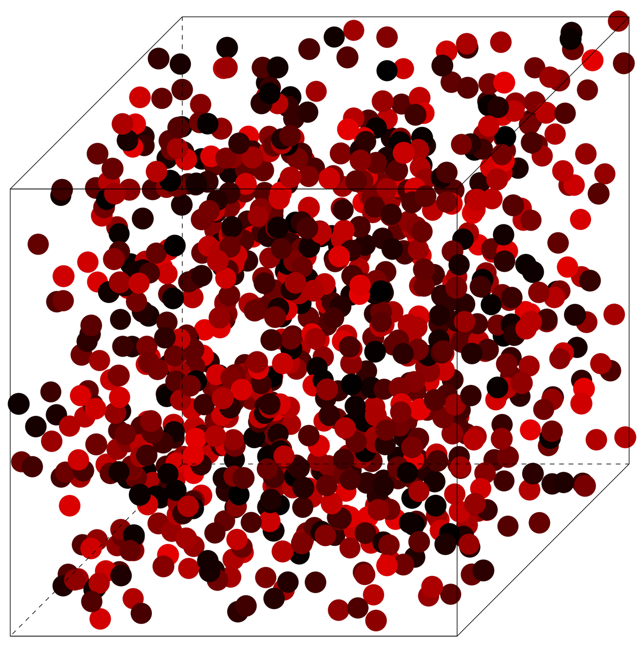

**Fig 2.** : Multiscale Material

## Building An Effective Coefficient

**Strategy For Best Effective Coefficient**

We examine the worst case scenario and try to minimize it upon $\overline{A}$.

$$I_\varepsilon = \inf_{\substack{\overline{A} \in \mathbb{R}^{d\times d}_{\text{sym}},\\ \alpha \le \overline{A} \le \beta}} J_\varepsilon(\overline{A}),$$

with

$$J_\varepsilon(\overline{A}) = \sup_{\substack{g \in L^2(\partial\mathcal{D}),\\ \|g\|_{L^2(\partial\mathcal{D})} = 1.}} \left( \left| \mathcal{O}(A_\varepsilon, g) - \mathcal{O}(\overline{A}, g) \right| \right).$$

The argInf $\overline{A}^{\text{opt}}_\varepsilon \in \mathbb{R}^{d\times d}_{\text{sym}}$ gives a satisfying effective description of the system : **the related solutions $\overline{u}$ are good $L^2$ approximation of $u_\varepsilon$.**

**Numerical Aspects :**

- **Cost Function** : The $\sup$ is replaced by a $\max$ over the space spanned by a few loadings $g_1, ..., g_P$ (in practice $P \approx 3$). -
- **Computational Cost** : Each step requires solving a $P$ *coarse* PDE. Using a gradient descent with adaptative step size (e.g. Armijo Rule) We need $N_{\text{iter}} \approx 15$ iterations.

## Comparison To Other Strategies

Our strategy is consistent with Homogenization theory (see [1]) in the sense that

**Consistency with Homogenization Theory**

In the **periodic setting** (i.e. $A_\varepsilon(x) = A_{\text{per}}(\frac{x}{\varepsilon})$), any sequence of *quasiminimizers* $(\overline{A}^{\#}_\varepsilon)_{\varepsilon>0}$ that satisfies $I_\varepsilon \le J_\varepsilon(\overline{A}^{\#}_\varepsilon) \le I_\varepsilon + \text{err}(\varepsilon)$ converges to the homogenized matrix $A_\star$ :

$$\lim_{\varepsilon \to 0} \overline{A}^{\#}_\varepsilon = A_\star.$$

In [2], another constant effective coefficient $\overline{B}^{\text{opt}}_\varepsilon \in \mathbb{R}^{d\times d}_{\text{sym}}$ is constructed using the knowledge of *full field measurements* $u_\varepsilon(g)$ : $\overline{B}^{\text{opt}}_\varepsilon = \text{arginf}_{\overline{A} \in \mathbb{R}^{d\times d}_{\text{sym}}} \sup_{g \in L^2(\partial\mathcal{D})} \|u_\varepsilon(g) - u(\overline{A}, g)\|_{L^2(\mathcal{D})} / \|g\|_{L^2(\partial\mathcal{D})}$.

We can compare the performance of both coefficients $\overline{A}^{\text{opt}}_\varepsilon$ and $\overline{B}^{\text{opt}}_\varepsilon$ using the criteria

$$\text{Err}(\overline{A}) = \sup_{g \in L^2(\partial\mathcal{D})} \frac{\|u_\varepsilon(g) - u(\overline{A}, g)\|_{L^2(\mathcal{D})}}{\|u_\varepsilon(g)\|_{L^2(\mathcal{D})}}.$$

**Using Complete Field Measurements**

There exists $C_1, C_2 > 0$ such that forany $\varepsilon > 0$,

$$\text{Err}(\overline{B}^{\text{opt}}_\varepsilon) \le C_1 \text{Err}(\overline{A}^{\text{opt}}_\varepsilon),$$
$$\text{Err}(\overline{A}^{\text{opt}}_\varepsilon) \le C_2 \sqrt{\text{Err}(\overline{B}^{\text{opt}}_\varepsilon)}.$$

### References

[1] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures.* Vol. 374. American Mathematical Soc., 2011.

[2] C. Le Bris, F. Legoll, and S. Lemaire. “On the best constant matrix approximating an oscillatory matrix-valued coefficient in divergence-form operators”. In: *ESAIM: Control, Optimisation and Calculus of Variations* 24.4 (2018), pp. 1345–1380.

## Numerical Results

Experiments have been conducted in 2D ($\mathcal{D} = [0,1]^2$) using the coefficient

$$A_{\text{per}}(x,y) = \begin{pmatrix} 22 + 10\,(\sin(2\pi x) + \sin(2\pi y)) & 0 \\ 0 & 12 + 2\,(\sin(2\pi x) + \sin(2\pi y)) \end{pmatrix},$$

and the loadings $g_{p,q}(x,y) = \cos(p\pi x)\cos(q\pi y)$.

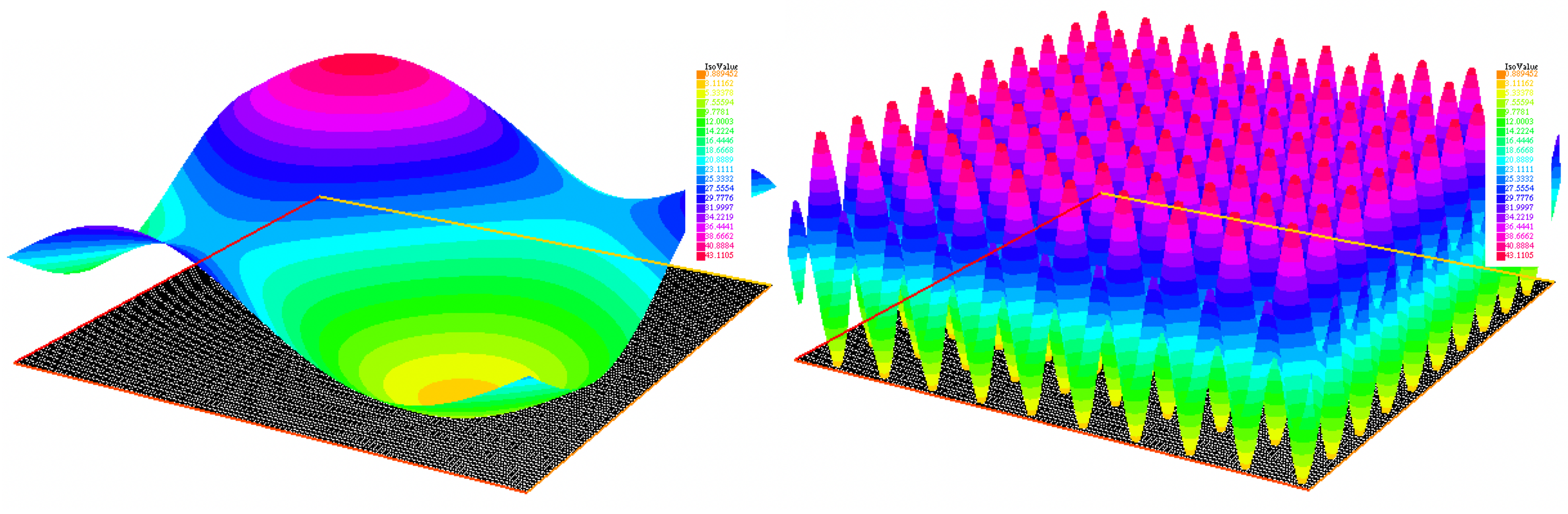

**Fig 3** : Kernel Coefficient $A_{\text{per},\,11}$ and Oscillating Coefficient $A_{\varepsilon,11} = A_{\text{per},\,11}\left(\frac{\cdot}{\varepsilon}\right)$ (with $\varepsilon = 0.1$).

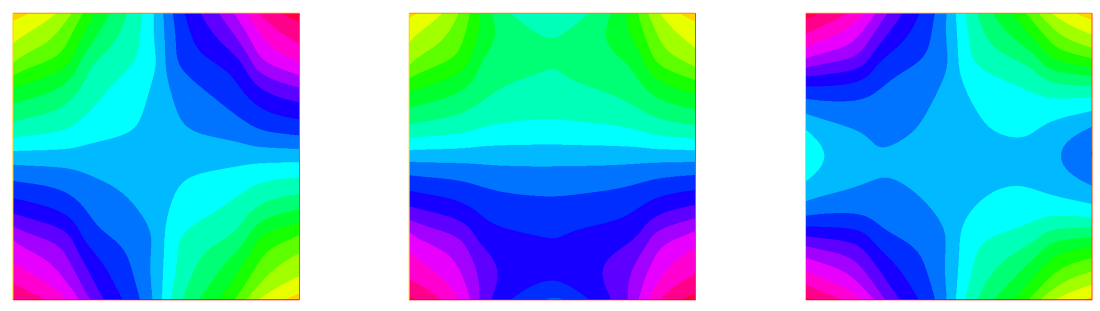

**Fig 4** : Loadings $g_{p,q}$.

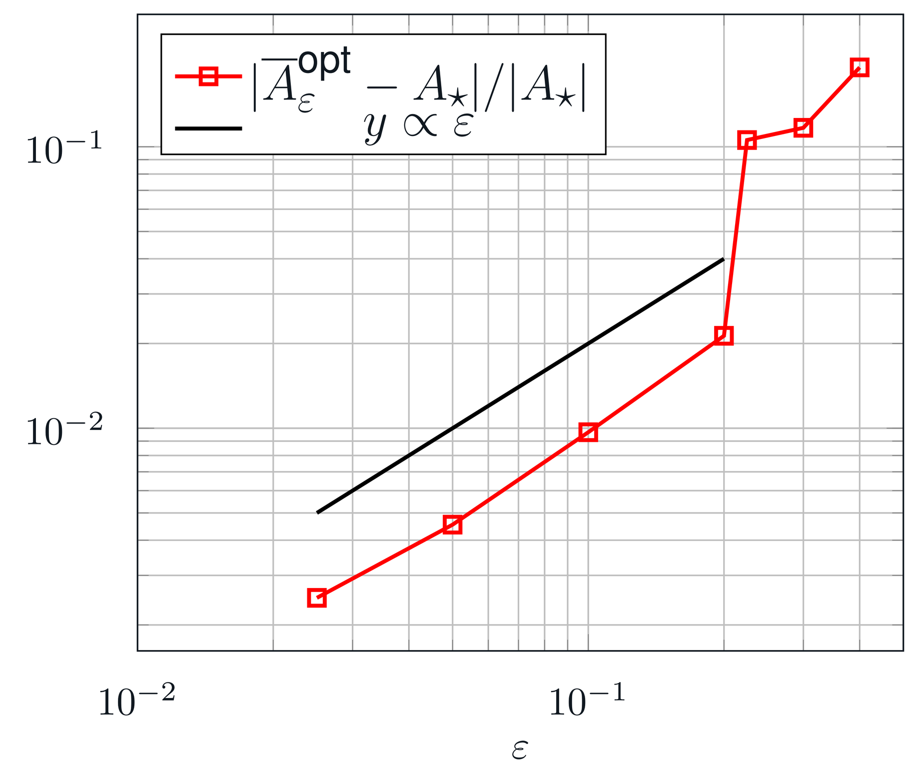


**Fig 5** : Error between homogenized coefficient $A_\star$ and effective coefficient $\overline{A}^{\text{opt}}_\varepsilon$ as a function of $\varepsilon$.

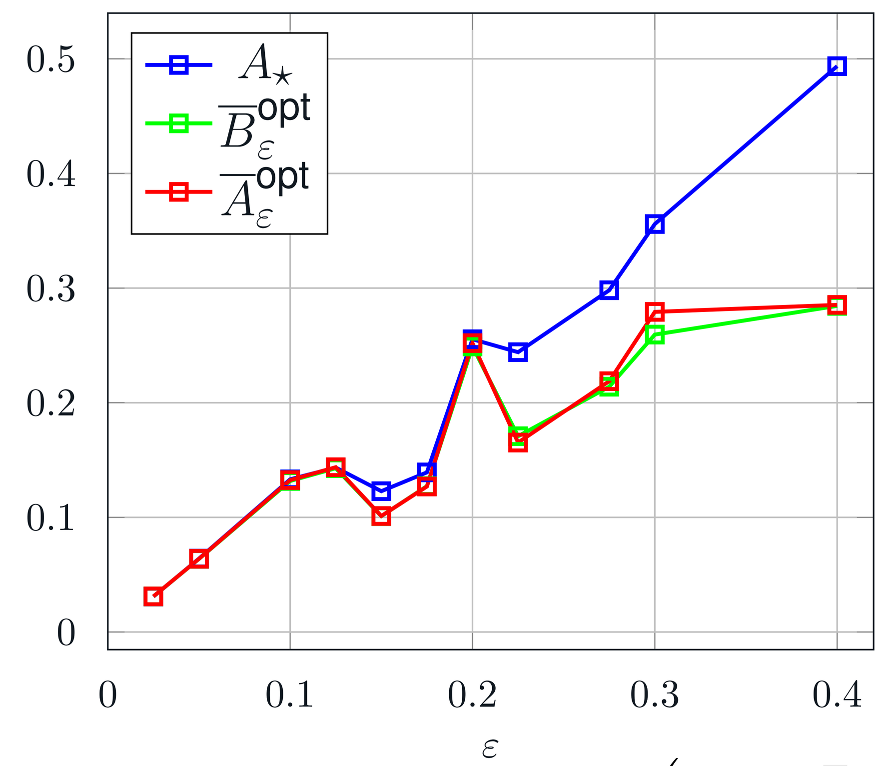


**Fig 6** : $L^2$ maximal error $\text{Err}_\varepsilon(\overline{A}) = \sup_{f \in \text{Span}_{1\le p\le 11}(g_p)} \left( \frac{\|u_\varepsilon(g) - \overline{u}(\overline{A}, g)\|_{L^2(\Omega)}}{\|u_\varepsilon(g)\|_{L^2(\Omega)}} \right)$ for various constant coefficient $\overline{A}$ ($\overline{A}^{\text{opt}}_\varepsilon$ and $\overline{B}^{\text{opt}}_\varepsilon$ computed using only $g_1, g_2$ and $g_3$).

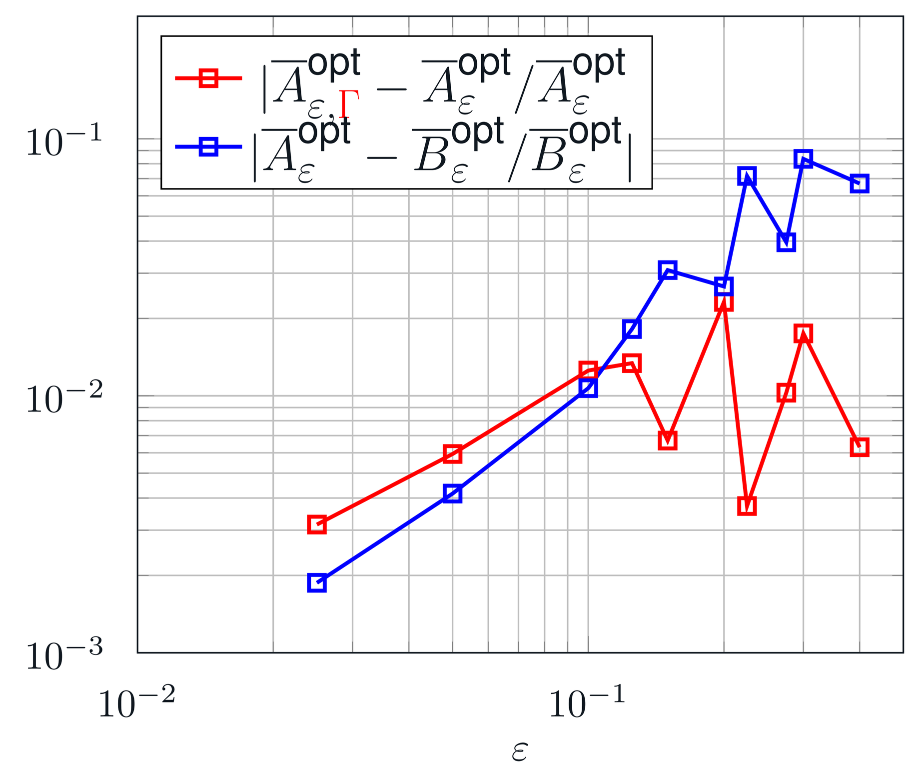


**Fig 7** : Relativ error between $\overline{A}^{\text{opt}}_\varepsilon$ and $\overline{A}^{\text{opt}}_{\varepsilon,\Gamma}$ (constructed using partial observables $\mathcal{O}_\Gamma(A_\varepsilon, g) = \int_\Gamma g u_\varepsilon(g)$ with $\Gamma \subsetneq \partial\mathcal{D}$ and $|\Gamma| = 0.5|\partial\mathcal{D}|$).