

# Cox Ross Rubinstein model

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## Abstract

We study the model introduced by Cox, Ross and Rubinstein in 1979 (see [1]). This report is widely inspired by [2]. In Section 1, we introduce the mathematical framework. In Section 2, we derive necessary and sufficient conditions for market viability and completeness. In Section 3, we focus on option pricing and hedging. In Section 4, we examine the relation to Black Scholes model.

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## 1 Framework

We consider two assets:

- a risk-free asset with interest rate  $r$  and initial price  $S_0^0 = 1$ . Its price at time  $0 \leq n \leq N$  is thus given by  $S_n^0 = (1 + r)^n$ .
- a risky asset, whose price at time  $0 \leq n \leq N$  is denoted  $S_n^1$ .

We assume that, between time  $n$  and time  $n + 1$ , the relative variation between the price of the risky asset is either  $a$  or  $b$ , with  $-1 < a < b$ . We write

$$S_{n+1}^1 = \begin{cases} S_n^1(1 + a) \\ S_n^1(1 + b) \end{cases} = S_n^1 T_{n+1}$$

where  $T_{n+1}$  takes value in  $\{1 + a, 1 + b\}$ . The initial price is denoted  $S_0^1$ .

Mathematically, the problem can be formalized as follows. We consider the space  $\Omega = \{1 + a, 1 + b\}^N$ . We consider the filtration defined by:  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  and, for any

$n \leq N$   $\mathcal{F}_n = \sigma(S_0^1, \dots, S_n^1)$  the  $\sigma$ -field generated by the random variables  $S_0^1, \dots, S_n^1$ . We endow  $(\Omega, \mathcal{P}(\Omega))$  with a probability measure  $\mathbb{P}$  such that each singleton of  $\Omega$  has a non null probability.

We recall that a probability measure  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  if and only if, for any event  $A \in \mathcal{P}(\Omega)$ ,  $\mathbb{P}(A) = 0 \iff \mathbb{Q}(A) = 0$ .

## 2 Market viability and completeness

**Proposition 1** (Necessary condition for market viability.). *If the market is viable, then it holds that*

$$\boxed{r \in ]a, b[.} \quad (1)$$

*Proof.* Let us assume that the market is viable. This means that there exists a probability  $\mathbb{P}^*$ , equivalent to  $\mathbb{P}$ , such that  $\tilde{\mathcal{S}}_n^1 = \frac{S_n^1}{S_0^1}$  is a martingale. This writes

$$\mathbb{E}^*(\tilde{\mathcal{S}}_{n+1}^1 | \mathcal{F}_n) = \tilde{\mathcal{S}}_n^1.$$

Observing that  $T_n = \frac{S_{n+1}^1}{S_n^1} = (1+r) \frac{\tilde{\mathcal{S}}_{n+1}^1}{\tilde{\mathcal{S}}_n^1}$ , it rewrites

$$\mathbb{E}^*((1+r)^{-1} \tilde{\mathcal{S}}_n^1 T_{n+1} | \mathcal{F}_n) = \tilde{\mathcal{S}}_n^1.$$

Since  $\tilde{\mathcal{S}}_n^1$  is  $\mathcal{F}_n$ -measurable, it implies that

$$\mathbb{E}^*(T_{n+1} | \mathcal{F}_n) = (1+r),$$

and, by taking the expectation, we deduce that

$$\mathbb{E}^*(T_{n+1}) = (1+r).$$

Since  $T_{n+1}$  takes value in  $\{1+a, 1+b\}$ , we have that

$$\mathbb{E}^*((1+a)\mathbb{1}_{T_{n+1}=1+a} + (1+b)\mathbb{1}_{T_{n+1}=1+b}) = (1+r).$$

By linearity, it holds that

$$(1+a)\mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+a}) + (1+b)\mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+b}) = (1+r).$$

Finally, since  $a < b$  and observing that  $\mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+a}) + \mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+b}) = 1$ , we conclude that

$$a < r < b.$$

□

**Remark 1** (Arbitrage when (1) is not satisfied). *When the condition (1) is not satisfied, we can easily identify arbitrage strategy.*

*Assume e.g. that  $r \leq a$ , and consider the following portfolio decomposition. At time 0, we borrow  $\phi_0^0$  unit of risk-free asset and buy  $\phi_1^0 = \frac{\phi_0^0}{S_0^1}$  unit of risky asset. At any time  $0 \leq n \leq N$ , we keep the same repartition of asset in the portfolio.*

*Clearly, this portfolio strategy is admissible and self-financing.*

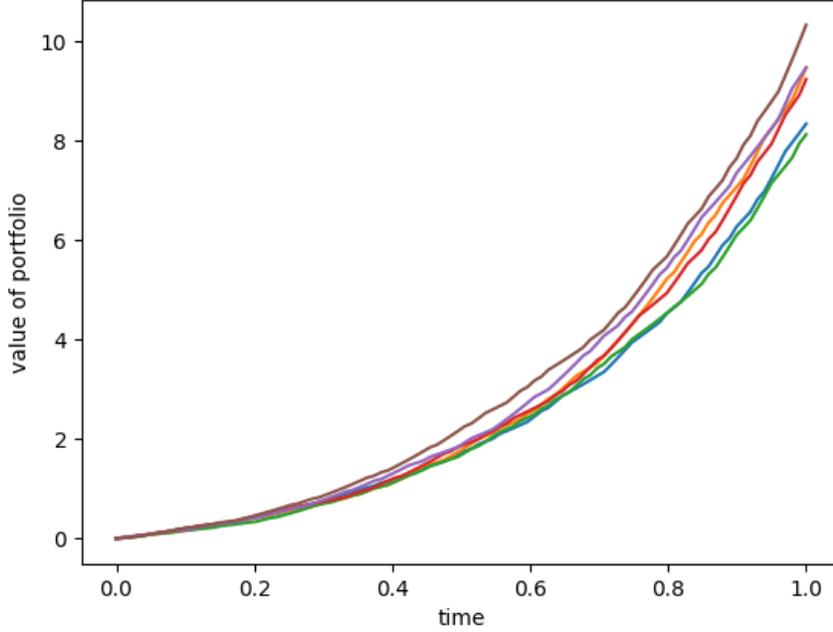


Figure 1: Existence of arbitrage when (1) is not satisfied. At any time, the value of the portfolio is positive.

*This is an arbitrage. At time  $N$ , we sell the risky asset and we reimburse the loan. The value of the portfolio at time  $N$ , is positive :*

$$\begin{aligned}
 V_N(\phi) &= -\phi_0^0(1+r)^N + \frac{\phi_0^0}{S_0^1}(1+a)^k(1+b)^{N-k} \\
 &\geq -\phi_0^0(1+r)^N + \frac{\phi_0^0}{S_0^1}(1+a)^N \\
 &> [-\phi_0^0 + \frac{\phi_0^0}{S_0^1}](1+r)^N \\
 &= 0.
 \end{aligned}$$

*Figure 1 shows the evolution of the portfolio's value for various market realization. At any time, the value of the portfolio is positive.*

**Proposition 2** (Necessary and sufficient conditions for market viability). *The price  $\tilde{S}_n^1$  is a  $\mathbb{P}^*$ -martingale if and only if the random variables  $T_1, \dots, T_N$  are i.i.d. and their common law is given by  $\mathbb{P}^*(T_1 = 1+a) = p$  and  $\mathbb{P}^*(T_1 = 1+b) = 1-p$  with  $p = \frac{b-r}{b-a}$  and  $r \in ]a, b[$ .*

*Proof.*  $\Rightarrow$  Let us assume that the price  $\tilde{S}_n^1$  is a  $\mathbb{P}^*$ -martingale . Then, using Proposition (1), we know that  $r \in ]a, b[$ .

Let us prove that  $T_1, \dots, T_N$  are independent and equidistributed. We proceed by recursion and show that

$$\mathbb{P}^*(T_1 = x_1, \dots, T_n = x_n) = \prod_{i=1}^n \mathbb{P}^*(T_i = x_i),$$

for any  $(x_i)_{1 \leq i \leq n} \in \{1+a, 1+b\}^n$ .

First, following calculations in the Proof of Proposition 1, we get that

$$\begin{cases} (1+a)\mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+a}|\mathcal{F}_n) + (1+b)\mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+b}|\mathcal{F}_n) = (1+r), \\ \mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+a}|\mathcal{F}_n) + \mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+b}|\mathcal{F}_n) = 1. \end{cases}$$

This directly implies that

$$\begin{cases} \mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+a}|\mathcal{F}_n) = p, \\ \mathbb{E}^*(\mathbb{1}_{T_{n+1}=1+b}|\mathcal{F}_n) = 1-p, \end{cases} \quad (2)$$

with  $p = \frac{b-r}{b-a}$ .

**Initialization.** Straightforward.

**Heredity.** Assume this equality holds for  $1 \leq n < N$ . For any  $(x_i)_{1 \leq i \leq n+1} \in \{1+a, 1+b\}^{n+1}$ , we have

$$\mathbb{P}^*(T_1 = x_1, \dots, T_{n+1} = x_{n+1}) = \mathbb{E}^*(\mathbb{1}_{T_{n+1}=x_{n+1}} \mathbb{1}_{T_1=x_1, \dots, T_n=x_n}).$$

The event  $(T_1 = x_1, \dots, T_n = x_n)$  belong to  $\mathcal{F}_n$ . Thus, by definition of conditional expectation, we have

$$\mathbb{P}^*(T_1 = x_1, \dots, T_{n+1} = x_{n+1}) = \mathbb{E}^*(\mathbb{E}^*(\mathbb{1}_{T_{n+1}=x_{n+1}}|\mathcal{F}_n) \mathbb{1}_{T_1=x_1, \dots, T_n=x_n}).$$

Using (2) and the recursion relation, we conclude that

$$\mathbb{P}^*(T_1 = x_1, \dots, T_n = x_n) = \prod_{i=1}^n \mathbb{P}^*(T_i = x_i),$$

$\Leftarrow$  Let us assume that the random variables  $T_1, \dots, T_N$  are i.i.d. and their common law is given by  $\mathbb{P}^*(T_1 = 1+a) = p$  and  $\mathbb{P}^*(T_1 = 1+b) = 1-p$  with  $p = \frac{b-r}{b-a}$ .

Then, we get that

$$\begin{aligned} \mathbb{E}^*(\tilde{\mathcal{S}}_{n+1}^1|\mathcal{F}_n) &= \mathbb{E}^*\left(\frac{T_{n+1}\mathcal{S}_n^1}{(1+r)^{n+1}}|\mathcal{F}_n\right) \\ &= \frac{\mathcal{S}_n^1}{(1+r)^{n+1}}\mathbb{E}^*(T_{n+1}|\mathcal{F}_n) \\ &= \frac{\mathcal{S}_n^1}{(1+r)^{n+1}}\mathbb{E}^*(T_{n+1}) \\ &= \frac{\mathcal{S}_n^1}{(1+r)^{n+1}}(1+r) \\ &= \frac{\mathcal{S}_n^1}{(1+r)^n} \\ &= \tilde{\mathcal{S}}_n^1. \end{aligned}$$

At the second line, we have used the fact that  $\mathcal{S}_n^1$  is  $\mathcal{F}_n$ -measurable. At the third, line, we have used the independence of  $T_{n+1}$  with respect to  $(T_k)_{1 \leq k \leq n}$  to get that  $\mathbb{E}^*(T_{n+1}|\mathcal{F}_n) = \mathbb{E}^*(T_{n+1})$ . At the fourth line, we have used the law of  $T_1$  to compute the value of the latter expectation.

We conclude that  $\tilde{\mathcal{S}}_n$  is a  $\mathbb{P}^*$ -martingale and, consequently, that the market is viable.  $\square$

**Remark 2** (Market completeness). *Proposition 2 reveals that the imposing the condition that the price  $\tilde{S}_n^1$  is a  $\mathbb{P}^*$ -martingale uniquely determines the law of  $(T_1, \dots, T_N)$ , and thus uniquely determines the probability  $\mathbb{P}^*$  itself. Market viability thus implies market completeness.*

### 3 Pricing and hedging

We assume that the hypothesis of Proposition 2 are satisfied.

We denote by  $C_n$  (resp.  $P_n$ ) the value at time  $n$  of a call (resp. put) european option on a unit of risky asset  $S_n$ , with strike  $K$  and maturity  $N$ .

**Proposition 3** (Call-Put parity relation). *It holds that*

$$\boxed{C_n - P_n = S_n - K(1+r)^{-(N-n)}} \quad (3)$$

*Proof.* Let us first recall that

$$\begin{aligned} C_n &= (1+r)^{-(N-n)} \mathbb{E}^*((S_N - K)_+ | \mathcal{F}_n), \\ P_n &= (1+r)^{-(N-n)} \mathbb{E}^*((K - S_N)_+ | \mathcal{F}_n). \end{aligned}$$

Thus, we get that

$$\begin{aligned} C_n - P_n &= (1+r)^{-(N-n)} \mathbb{E}^*((S_N - K)_+ - (K - S_N)_+ | \mathcal{F}_n) \\ &= (1+r)^{-(N-n)} \mathbb{E}^*(S_N - K | \mathcal{F}_n) \\ &= (1+r)^{-(N-n)} \mathbb{E}^*(S_N | \mathcal{F}_n) - K(1+r)^{-(N-n)} \\ &= S_n - K(1+r)^{-(N-n)}. \end{aligned}$$

where we used that  $\tilde{S}_n = \frac{S_n}{(1+r)^n}$  is a martingale under  $\mathbb{P}^*$ . □

Figure 2 shows that the call put parity relation is indeed satisfied.

**Proposition 4** (Formula for call). *The value of a call option at time  $n$  writes*

$$\boxed{C_n = c(n, S_n)}, \quad (4)$$

where  $c$  satisfies the recursive relation

$$\boxed{c(n, s) = \frac{pc(n+1, s(1+a)) + (1-p)c(n+1, s(1+b))}{1+r}}. \quad (5)$$

The function  $c$  depends only on  $K, a, b, r, p$  and writes

$$c(n, s) = (1+r)^{-(N-n)} \sum_{j=0}^{N-n} C_{N-n}^j p^j (1-p)^{N-n-j} (s(1+a)^j (1+b)^{N-n-j} - K)_+. \quad (6)$$

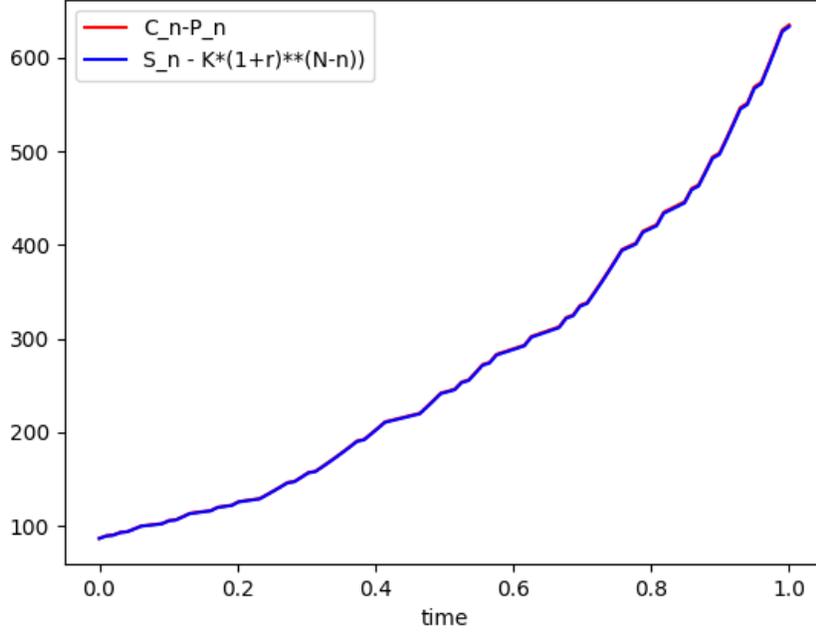


Figure 2: Call Put Parity Relation. We consider  $a = 0.01, b = 0.03, r = 0.02, K = 95, S_0^1 = 100$ .

*Proof.* We recall that

$$C_n = (1 + r)^{-(N-n)} \mathbb{E}^* ((S_N - K) | \mathcal{F}_n)$$

Observing that  $S_N = S_n \prod_{j=n+1}^N T_j$ , we get

$$C_n = (1 + r)^{-(N-n)} \mathbb{E}^* ((S_n \prod_{j=n+1}^N T_j - K) | \mathcal{F}_n)$$

Since  $S_n$  is  $\mathcal{F}_n$ -measurable and  $(T_j)_{j \geq n+1}$  are  $\mathcal{F}_n$ -independent, we have

$$\mathbb{E}^* ((S_n \prod_{j=n+1}^N T_j - K) | \mathcal{F}_n) = \phi(S_n),$$

where

$$\begin{aligned} \phi(s) &= \mathbb{E}^* \left( (s \prod_{j=n+1}^N T_j - K)_+ \right) \\ &= \sum_{j=0}^{N-n} C_{N-n}^j p^j (1-p)^{N-n-j} (s(1+a)^j (1+b)^{N-n-j} - K)_+ \end{aligned}$$

Finally, we conclude that

$$C_n = c(n, S_n)$$

with

$$c(n, s) = (1 + r)^{-(N-n)} \sum_{j=0}^{N-n} C_{N-n}^j p^j (1-p)^{N-n-j} (s(1+a)^j (1+b)^{N-n-j} - K)_+$$

To obtain the recursive relation, we condition according to  $T_{n+1}$  which is equal to  $1 + a$  with probability  $p$  and to  $1 + b$  with probability  $1 - p$ . It holds that

$$\begin{aligned}\phi(s) &= \mathbb{E}^* \left( (s \Pi_{j=n+1}^N T_j - K)_+ \right) \\ &= p \mathbb{E}^* \left( (s \Pi_{j=n+1}^N T_j - K)_+ | T_{n+1} = 1 + a \right) + (1 - p) \mathbb{E}^* \left( (s \Pi_{j=n+1}^N T_j - K)_+ | T_{n+1} = 1 + b \right) \\ &= p \mathbb{E}^* \left( (s(1 + a) \Pi_{j=n+2}^N T_j - K)_+ \right) + (1 - p) \mathbb{E}^* \left( (s(1 + b) \Pi_{j=n+2}^N T_j - K)_+ \right)\end{aligned}$$

Multiplying by  $(1 + r)^{-(N-n)}$ , we get the relation 6.  $\square$

**Proposition 5** (Hedging a call option). *The best hedging strategy for a call consists in detaining a quantity  $H_n^0 = \Delta^0(n, S_{n-1})$  of risk-free asset and a quantity  $H_n^1 = \Delta^1(n, S_{n-1})$  of risky asset, with  $\Delta^0$  and  $\Delta^1$  given by*

$$\begin{cases} \Delta^0(n, s) = \frac{(1 + b)c(n, x(1 + a)) - (1 + a)c(n, x(1 + b))}{(1 + r)^n(b - a)} \\ \Delta^1(n, s) = \frac{c(n, x(1 + b)) - c(n, x(1 + a))}{s(b - a)} \end{cases} \quad (7)$$

*Proof.* The best hedging strategy to hedge for a call is defined as follows. At time  $n$ , we must detain a quantity  $H_n^0$  of risk-free asset and a quantity  $H_n^1$  of risk-asset such that

$$H_n^0(1 + r)^n + H_n^1 S_n = c(n, S_n).$$

In particular, since  $S_n = S_{n-1} T_n$ , this equation implies that

$$\begin{cases} H_n^0(1 + r)^n + H_n^1 S_{n-1}(1 + a) = c(n, S_{n-1}(1 + a)), \\ H_n^0(1 + r)^n + H_n^1 S_{n-1}(1 + b) = c(n, S_{n-1}(1 + b)). \end{cases}$$

Simplifying this system leads to the expression of  $\Delta^0$  and  $\Delta^1$ .  $\square$

## 4 Relation to Black-Scholes model

In this section, we show that the relation between Cox Ross Rubinstein model and the Black Scholes model. More precisely, we show that the discrete CRR model converges to the continuous BS model when the number of periods tends to be very large. We recover the value of call and put options established by the BS formulas.

Let us consider a call of maturity  $T$ . We divide  $[0, T]$  into  $N$  intervals. We impose that

$$\begin{aligned}r &= \frac{RT}{N} \\ \log \left( \frac{1 + a}{1 + r} \right) &= -\sigma \sqrt{\frac{T}{N}} \\ \log \left( \frac{1 + b}{1 + r} \right) &= \sigma \sqrt{\frac{T}{N}}\end{aligned}$$

where  $R$  represents an instantaneous interest rate (since  $e^{RT} = \lim_{N \rightarrow +\infty} (1+r)^N$ ). The number  $\sigma^2 T$  can be interpreted as the limit variance of the random variable  $\log(S_N)$ .

Let us fix  $N$  and consider the value  $C_N(K, T)$  of a call. We have

$$\begin{aligned} C_N(K, T) &= \left(1 + \frac{RT}{N}\right)^{-N} \mathbb{E}^* \left( (S_N - K)_+ \right) \\ &= \left(1 + \frac{RT}{N}\right)^{-N} \mathbb{E}^* \left( (S_0 \prod_{j=1}^N T_j - K)_+ \right) \\ &= \mathbb{E}^* \left( \left( S_0 \exp(Y_N) - K \left(1 + \frac{RT}{N}\right)^{-N} \right)_+ \right), \end{aligned}$$

with

$$Y_N = \sum_{j=1}^N \log \left( \frac{T_j}{1+r} \right).$$

It can be shown that  $Y_N$  converges in law to a gaussian  $\mathcal{N}(-\sigma^2 T/2, \sigma^2 T)$ .

Let us introduce  $\psi(y) = (S_0 e^y - K e^{-RT})_+$ . We have

$$\begin{aligned} |C_N(K, T) - \mathbb{E}^*(\psi(Y_N))| &= \left| \mathbb{E}^* \left( \left( S_0 \exp(Y_N) - K \left(1 + \frac{RT}{N}\right)^{-N} \right)_+ - (S_0 e^{Y_N} - K e^{-RT})_+ \right) \right| \\ &\leq K \left| e^{-RT} - \left(1 + \frac{RT}{N}\right)^{-N} \right| \\ &\xrightarrow{N \rightarrow 0} 0 \end{aligned}$$

Since  $\psi$  is continuous and bounded, we deduce that

$$\begin{aligned} \lim_{N \rightarrow 0} C_N(K, T) &= \lim_{N \rightarrow 0} \mathbb{E}^*(\psi(Y_N)) \\ &= \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{+\infty} (S_0 e^y - K e^{-RT})_+ e^{-\frac{(y+\sigma^2 T/2)^2}{2\sigma^2 T}} dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (S_0 e^{\sigma\sqrt{T}y - \sigma^2 T/2} - K e^{-RT})_+ e^{-\frac{y^2}{2}} dy. \end{aligned}$$

Finally, we recover the Black-Scholes formula for the value of a call option.

$$\boxed{\lim_{N \rightarrow 0} C_N(K, T) = S_0 \phi(-d_2) - K e^{-RT} \phi(-d_1)}, \quad (8)$$

where

$$\begin{cases} d_1 = \frac{1}{\sigma\sqrt{T}} \left( \frac{\sigma^2 T}{2} + \log \left( \frac{K}{S_0} \right) - RT \right), \\ d_2 = d_1 - \sigma\sqrt{T}, \\ \phi(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^d e^{-x^2/2} dx. \end{cases}$$

## References

- [1] J. C. Cox, S. A. Ross, and M. Rubinstein. “Option pricing: A simplified approach”. In: *Journal of Financial Economics* 7.3 (1979), pp. 229–263.
- [2] D. Lamberton and B. Lapeyre. *Introduction to stochastic calculus applied to finance*. Chapman and Hall, 1996.