Construction of coarse approximations for problems with highly oscillating coefficients

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An Inverse Multiscale Problem

Our study focuses on multiscale systems (e.g. composite materials).

Determining the fine-scale structure from measurements is an ill-posed inverse problem (see [Lions78][1]) but identifying effective parameters is possible !

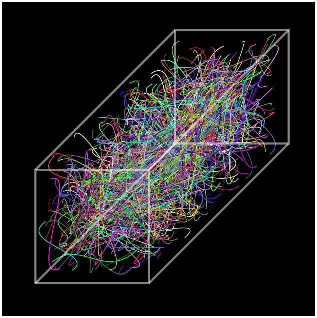


Figure: Scheme of a composite material composed of microfibers randomly intertwined.

[1] J.-L. Lions, *Some aspects of modelling problems in distributed parameter systems*, 1978.

Multiscale Context : ill-posed inverse problem

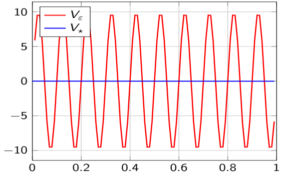
Consider the prototypical linear equation with caracteristic length scall ε

$$\mathcal{L}_\varepsilon u_\varepsilon = (-\Delta + \varepsilon^{-1} V(\varepsilon^{-1}\cdot)) u_\varepsilon = f \text{ in } \Omega, \qquad u_\varepsilon = 0 \text{ on } \partial\Omega,$$

with **periodic** coefficient V such that $\langle V \rangle = 0$, and RHS $f \in L^2(\Omega)$.
Homogenization theory[2] assesses the existence of a limit equation when $\varepsilon \to 0$:

$$\mathcal{L}_\star u_\star = (-\Delta + V_\star) u_\star = f \text{ in } \Omega, \qquad u_\star = 0 \text{ on } \partial\Omega,$$

and we have $u_\varepsilon \to u_\star$ strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega)$ when $\varepsilon \to 0$.



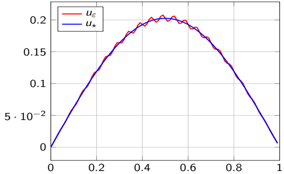


Figure: Two similar solutions associated to two very distinct coefficients.

[2] A. Bensoussan, J.-L. Lions, G. Papanicolaou, *Asymptotic analysis for periodic structures*, 1978

Our Approach

Consider the problem (1) involving a periodic coefficient V :

$$\mathcal{L}_\varepsilon u_\varepsilon = \left(-\Delta + \varepsilon^{-1} V\left(\varepsilon^{-1}\cdot\right)\right) u_\varepsilon = f \text{ in } \Omega, \qquad u_\varepsilon = 0 \text{ on } \partial\Omega. \tag{1}$$

From the knowledge of solutions u_ε for various RHS f, our aim is to propose a numerical methodology to build an effective operator $\overline{\mathcal{L}}$ approaching $\mathcal{L}_\varepsilon$ with satisfying L^2 error on the solutions.

Our strategy

- is inspired by homogenization theory,
- does not rely at its core on classical hypothesis for homogenization (such as periodicity), which may be too restrictive in practical situations,
- is valid outside the regime of homogenization (i.e. $\varepsilon \to 0$),
- can be adapted to a wide range of other equations (see [LeBris18][3]).

[3] C. Le Bris, F. Legoll, and S. Lemaire, *On the best constant matrix approximating an oscillatory matrix-valued coefficient in divergence-form operators*, 2018.

Recovering an effective coefficient

Let $\overline{V} \in \mathbb{R}$ be a *constant* coefficient, and $\overline{u} = u(\overline{V}, f)$ be the solution to (2) with RHS $f \in L^2(\Omega)$.

$$\overline{\mathcal{L}}\overline{u} = \left(-\Delta + \overline{V}\right)\overline{u} = f \text{ in } \Omega, \qquad \overline{u} = 0 \text{ on } \partial\Omega. \tag{2}$$

The quality of the approximation of $\mathcal{L}_\varepsilon$ by $\overline{\mathcal{L}}$ can be quantified through the functional

$$\sup_{\|f\|_{L^2(\Omega)}=1} \left\| u_\varepsilon(f) - u(\overline{V}, f) \right\|^2_{L^2(\Omega)}$$

Hence we can minimize the worst case scenario with the optimization problem

$$I_\varepsilon = \inf_{\overline{V}\in\mathbb{R}} \sup_{\|f\|_{L^2(\Omega)}=1} \left\| u_\varepsilon(f) - u(\overline{V}, f) \right\|^2_{L^2(\Omega)} \tag{3}$$

The choice of an $L^2(\Omega)$ norm is reminescent of the fact that $\|u_\varepsilon - u_\star\|_{L^2(\Omega)}$ tends to 0 with ε.

(In practice, we slighty modify (3) to recover a quadratic problem.)

Consistency with homogenization in the regime of separated scales

The problem

$$I_\varepsilon = \inf_{\overline{V}\in\mathbb{R}} \sup_{\|f\|_{L^2(\Omega)}=1} \|u_\varepsilon(f) - u(\overline{V}, f)\|^2_{L^2(\Omega)} \tag{3}$$

is consistent with homogenization theory in the sense that :

$$\lim_{\varepsilon\to 0} I_\varepsilon = 0. \tag{4}$$

Furthermore, for $\varepsilon > 0$ fixed (sufficiently small), there exists a unique minimizer $\overline{V}^0_\varepsilon \in \mathbb{R}$. The following convergence holds :

$$\boxed{\lim_{\varepsilon\to 0} \overline{V}^0_\varepsilon = V_\star,} \tag{5}$$

where $V_\star$ is the homogenized coefficient.

Numerical results

To solve

$$I_\varepsilon = \inf_{\overline{V}\in\mathbb{R}} \sup_{\|f\|_{L^2(\Omega)}=1} \|u_\varepsilon(f) - u(\overline{V}, f)\|^2_{L^2(\Omega)},$$

we use an Uzawa algorithm (i.e. $\min_{\overline{V}}$ intertwinned with $\max_f$) in 2D ($\Omega = [0,1]^2$) using the coefficient

$$V(x,y) = c\left(\sin(2\pi x) + \sin(2\pi y)\right).$$

We approximate the supremum by a maximization over the first eigenmodes of $(-\Delta)$-operator. In practice, a *single* mode is often sufficient in order to find the *single* coefficient $\overline{V}$.

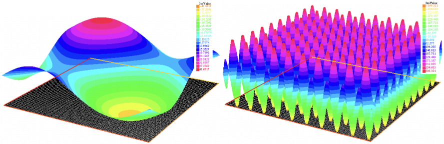


Figure: Coefficient V and Oscillating Coefficient $V_\varepsilon = \varepsilon^{-1} V(\varepsilon^{-1}\cdot)$.

Relationship with homogenized potential

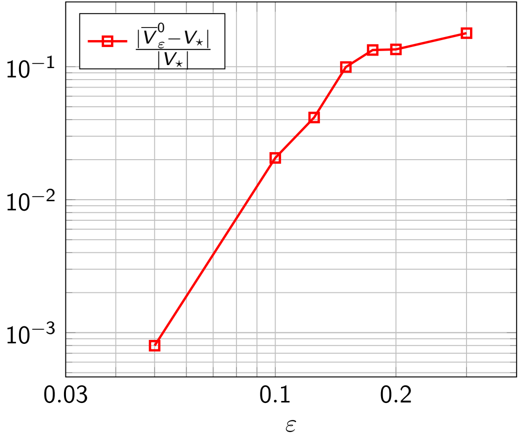


Figure: Error between the homogenized coefficient $V_\star$ and the effective coefficient $\overline{V}_\varepsilon^0$ as a function of ε.

Beyond the regime of separated scales

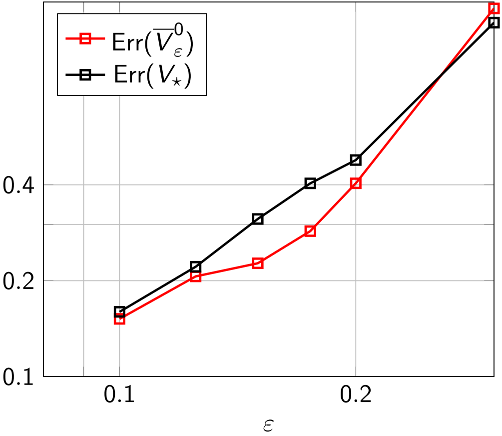


Figure: Error $\mathrm{Err}(\mathbf{V}) = \sup_{f \in \mathrm{Span}_{1\leq p\leq 10}(f_p)} \left(\frac{\|u_\varepsilon(f) - u(\mathbf{V}, f)\|_{L^2(\Omega)}}{\|u_\varepsilon(f)\|_{L^2(\Omega)}} \right)$ as a function of ε for $\mathbf{V} = V_\star$ and $\mathbf{V} = \overline{V}_\varepsilon^0$.

Future Work

In progress :

- Exploring other type of data : **boundary measurements**, **macroscopic data** $\left(\text{e.g. energy } \int_\Omega |\nabla u_\varepsilon|^2 + V_\varepsilon u_\varepsilon^2\right)$ instead of microscopic ones, ...
- Robustness analysis : what happens if the data is blurred/perturbed/deteriorated/... ?
- Exploring pertubative setting.
- Performing classification rather than identification of the effective coefficient.

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An Inverse Multiscale Problem

How to tackle such problems ?

General approach :

- **Direct Inversion** (see [Uhlmann13]),
- **Perturbation** (see [Ammari08]),

Approach based on homogenization (see [Nolen12]):

- **Features Determination** (see [Engquist14]),
- **Regularization at order** 0, identifying effective quantity (see [Ammari16, Caiazzo20]),
- **Regularization at order** 1, beyond effective quantity : H^1 reconstruction (see [Garnier23, LeBris18]).
- **Inverse Homogenization** (see [Cherkaev01])