Effective approximations of multiscale PDE based on limited information

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General introduction

Multiscale systems

Multiscale systems are characterized by the presence of several scales of interest that interact or influence one another.

They may be found in various scientific areas: engineering, biology, physics, ...

airplane wing ≈ 10m v.s. carbon fibers $\approx 10^{-6}$m

Figure: Composite material used in the aeronautics industry.

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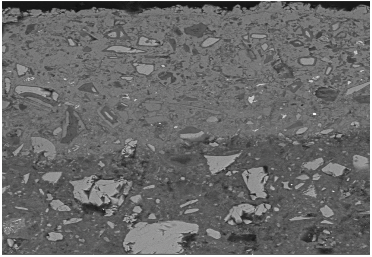

bridge $\approx 10^3$m v.s. mineral aggregate $\approx 10^{-5}$m

Figure: Concrete: a multiscale material.

Approximation of multiscale PDE

- Consider the problem
$$-\mathrm{div}\left(A_\varepsilon \nabla u_\varepsilon\right) = f \text{ in } \Omega,$$
where A_ε is oscillating at a small length scale $\varepsilon \ll |\Omega|$.

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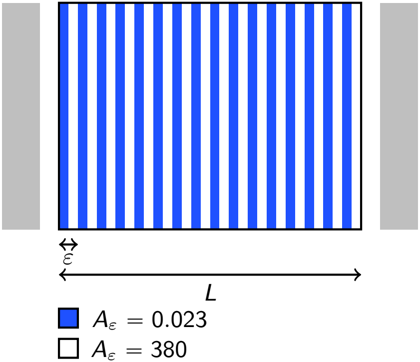
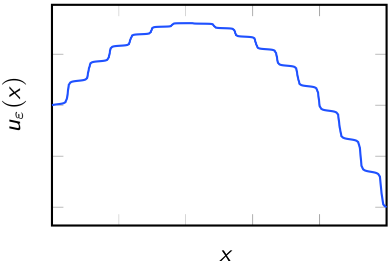
- Applications: heat transfer in thermal engineering, (simplification of) elastic problem in mechanics, ...





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where A_ε is oscillating at a small length scale $\varepsilon \ll |\Omega|$.

Objective

Based on measurements about the system, construct an approximation of the mapping
$$\mathcal{L}_\varepsilon : f \to u_\varepsilon(f).$$

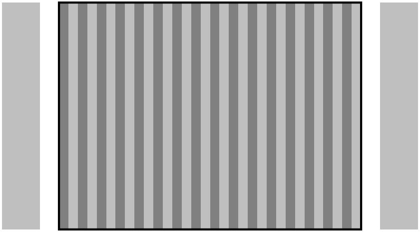
Limited information

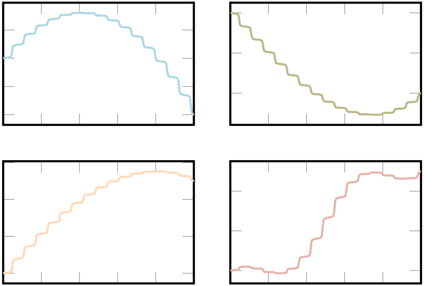
Experimental settings:

- little knowledge on microstructure.
- availability of couples $(\textit{configuration}, \textit{system response})$.

Settings with <u>limited</u> information:

- No assumptions on microstructure (non periodic case, ε small but not infinitely small, ...).
- Qualitative restrictions (coarse measurements, noisy measurements, ...).
- Quantitative restrictions (limited budget of measurements).





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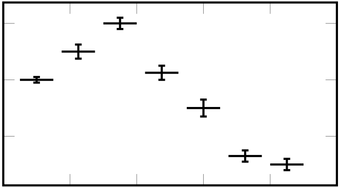

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Homogenization (see e.g. [BLP78][1]) builds PDE with slowly varying coefficients that accurately approximate the oscillating PDE.

$$\underbrace{\boxed{\begin{cases} -\mathrm{div}\,(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}}}_{\text{Oscillating System}} \xrightarrow[\varepsilon \to 0]{} \underbrace{\boxed{\begin{cases} -\mathrm{div}\,(A_\star \nabla u_\star) = f & \text{in } \Omega, \\ u_\star = 0 & \text{on } \partial\Omega. \end{cases}}}_{\text{Homogenized System}}$$

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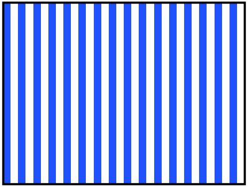

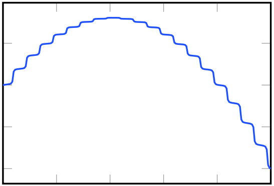

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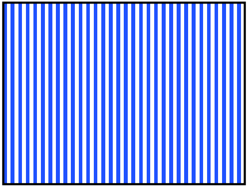

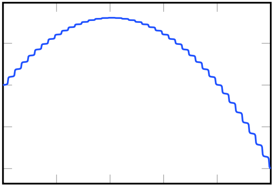

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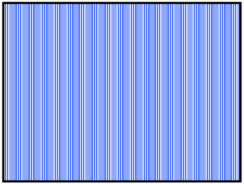

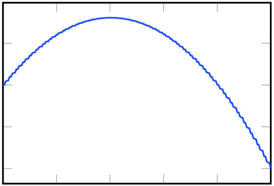

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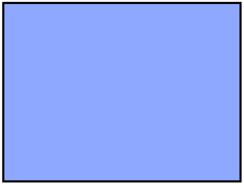

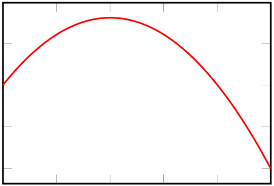

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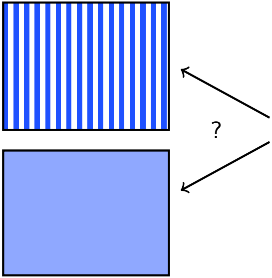

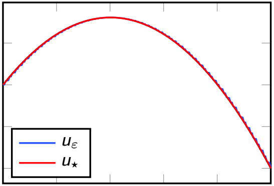

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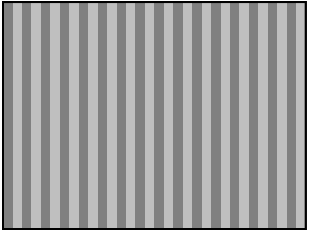

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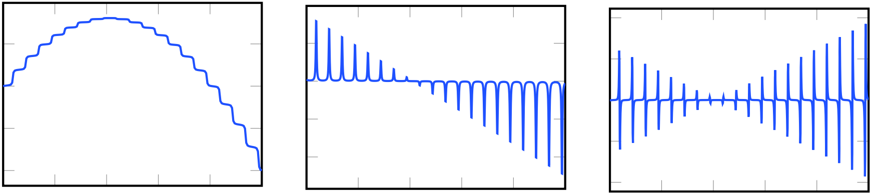
u_ε $\quad$ $\partial_x u_\varepsilon$ $\quad$ $\partial^2_{xx} u_\varepsilon$



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Issue: how to proceed in contexts of limited information?

Effective coefficients are coefficients varying at the macroscopic scale that encapsulate the fine-scale features of highly oscillatory coefficients.

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Example. Homogenization assesses the existence of an *effective* coefficient $A_\star$ such that

$$\mathcal{L}_\varepsilon : f \longrightarrow u_\varepsilon(f) \qquad \text{sol. to } -\mathrm{div}\left(A_\varepsilon \nabla u_\varepsilon\right) = f$$

$$\Big\downarrow \varepsilon \to 0$$

$$\mathcal{L}_{A_\star} : f \longrightarrow u_\star(f) \qquad \text{sol. to } -\mathrm{div}\left(A_\star \nabla u_\star\right) = f$$

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In the periodic case $A_\varepsilon(x) = A_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right)$, with A_{per} Q-periodic:

$$A_\star = \int_Q A_{\mathrm{per}}(\nabla w + \mathrm{Id}),$$

where w is a corrector defined through a PDE involving A_{per}.

In particular, in dimension $d = 1$, it holds that $A_\star = \left(\int_Q \frac{1}{A_{\mathrm{per}}}\right)^{-1}$.

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Two major limitations of homogenization:

- No formulas for $A_\star$ in the general case.
- Valid only in the regime of separated scale (i.e. $\varepsilon \to 0$).

Overview

Objective

Based on available observables, define an effective operator $-div\left(\overline{A}\nabla\cdot\right)$ *such that, for any* f*, the solutions* $u_\varepsilon(f)$ *to*

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are satisfactorily approximated by the solutions $\overline{u} = u(\overline{A}, f)$ *to the coarse problem*

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Part I · · · · · · Construct $\overline{A}$ in the set $\mathbb{R}^{d\times d}_{\mathrm{sym}}$.

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(**Part IV** ······ Effective approximation of Schrödinger equation).

Part I

Effective modeling from boundary agregated measurements

A proof of concept [CRAS2013][2], [COCV2018][3]

For any $g \in L^2_0(\partial\Omega)$, consider the solution $u_\varepsilon = u_\varepsilon(g)$ with vanishing mean to

$$-\operatorname{div}\left(A_\varepsilon \nabla u_\varepsilon\right) = 0 \text{ in } \Omega, \qquad \left(A_\varepsilon \nabla u_\varepsilon\right) \cdot n = g \text{ on } \partial\Omega. \tag{1}$$

[2] C. Le Bris, F. Legoll, K. Li, CRAS, 2013.

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The quality of the effective coefficient $\overline{A}$ can be quantified through the functional

$$\sup_{\|g\|_{L^2(\partial\Omega)}=1} \|u_\varepsilon(g) - u(\overline{A}, g)\|_{L^2(\Omega)}.$$

The strategy consists in minimizing the worst case scenario by looking at the optimization problem

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Issue: Using the **full solutions u_ε in the whole domain Ω** as observables is **disproportionate** to estimate a $d \times d$ constant symmetric matrix, and **irrealistic** from an experimental point of view.

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Practical observables

Only *coarser* observables are usually experimentally accessible, such as the energy

$$\boxed{\mathcal{E}(A_\varepsilon, g) = \frac{1}{2}\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon - \int_{\partial\Omega} g \;\, u_\varepsilon(g) = -\frac{1}{2}\int_{\partial\Omega} g \;\, u_\varepsilon(g).} \tag{3}$$

Discussions with E. Baranger, L. Chamoin, E. Daghia (LPMS, ENS Paris-Saclay).

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Motivation:

- $\mathcal{E}(A_\varepsilon, g)$ passes to the homogenized limit:

$$\mathcal{E}(A_\varepsilon, g) \underset{\varepsilon \to 0}{\longrightarrow} \mathcal{E}(A_\star, g) \text{ in } \mathbb{R},$$

  where $\mathcal{E}(A_\star, g) = \frac{1}{2}\int_\Omega A_\star \nabla u_\star \cdot \nabla u_\star - \int_{\partial\Omega} g \;\, u_\star$ and where $u_\star$ denotes the homogenized solution.
- $\mathcal{E}(A_\varepsilon, g)$ is an integrated quantity at the boundary, thus it presents the characteristics of a quantity that is experimentally accessible.
- $\mathcal{E}(A_\varepsilon, g)$ is a scalar, thus it provides no direct insights about the microscale.

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A new formulation

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$$\cancel{\sup_{\|g\|_{L^2(\partial\Omega)}=1} \left\|u_\varepsilon(g) - u(\overline{A}, g)\right\|_{L^2(\Omega)}^2} \longrightarrow \sup_{\|g\|_{L^2(\partial\Omega)}=1} \left|\mathcal{E}(A_\varepsilon, g) - \mathcal{E}(\overline{A}, g)\right|^2.$$

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Our strategy consists in minimizing the worst case scenario by looking at the optimization problem

$$\inf_{\substack{\overline{A}\in\mathbb{R}^{d\times d}_{\mathrm{sym}} \\ \alpha \leqslant \overline{A} \leqslant \beta}} \sup_{\|g\|_{L^2(\partial\Omega)}=1} \left|\mathcal{E}(A_\varepsilon, g) - \mathcal{E}(\overline{A}, g)\right|^2.$$

In the limit of vanishing ε, the problem leads to the homogenized diffusion coefficient as shown by the following proposition.

$$I_\varepsilon = \inf_{\substack{\overline{A} \in \mathbb{R}^{d\times d}_{\mathrm{sym}} \\ \alpha \leqslant \overline{A} \leqslant \beta}} \underbrace{\sup_{\|g\|_{L^2(\partial\Omega)}=1} \left|\mathcal{E}(A_\varepsilon, g) - \mathcal{E}(\overline{A}, g)\right|^2}_{J_\varepsilon(\overline{A})}$$

Proposition (Asymptotic consistency, periodic case)

For any sequence of quasi-minimizers $\left(\overline{A}^{\#}_\varepsilon\right)_{\varepsilon>0}$, *i.e. sequence such that*

$$I_\varepsilon \leqslant J_\varepsilon(\overline{A}^{\#}_\varepsilon) \leqslant I_\varepsilon + err(\varepsilon),$$

the following convergence holds:

$$\boxed{\lim_{\varepsilon\to 0} \overline{A}^{\#}_\varepsilon = A_\star.} \tag{4}$$

Sketch of proof

Three ingredients:

- Optimization over compact set $\mathcal{S}_{\alpha,\beta} \implies \overline{A}_\varepsilon^{\#}$ converges to $\overline{A}_{\#}$ up to an extraction.

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  Polarization relation implies that for any $f, g \in L^2_0(\partial\Omega)$:

  $$\int_{\partial\Omega} f \; u(A_\star, g) = \int_{\partial\Omega} f \; u(\overline{A}_\#, g).$$

  Thus,

  $$u(A_\star, g) = u(\overline{A}_\#, g) \text{ in } L^2(\partial\Omega). \tag{5}$$

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  Thus,

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- Use that $A_\star, \overline{A}_\#$ are constant coefficients and exploit (5) evaluated at particular loadings $(g_i)_{1\leqslant i\leqslant \frac{d(d+1)}{2}}$ to conclude that

  $$A_\star = A_\#.$$

Computational procedure

We apply an iterative algorithm to solve

$$\inf_{\substack{\overline{A} \in \mathbb{R}^{d\times d}_{\text{sym}} \\ \alpha \leqslant \overline{A} \leqslant \beta}} \sup_{\|g\|_{L^2(\partial\Omega)}=1} \left|\mathcal{E}(A_\varepsilon, g) - \mathcal{E}(\overline{A}, g)\right|^2.$$

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Given an iterate $\overline{A}^n$,

1. Define g^n, the argsup to

$$\sup_{g \ \text{ s.t. } \|g\|_{L^2(\partial\Omega)} = 1} \Big(\mathcal{E}(A_\varepsilon, g) - \mathcal{E}(\overline{A}^n, g)\Big)^2.$$

2. Define $\overline{A}^{n+1}$, the optimizer to

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   In practice, $\sup_{g\in L^2_0(\Omega)} \to \sup_{g\in V_P}$ on $V_P = \mathrm{Span}\{P \text{ loadings}\}$, with $P \approx \frac{d(d+1)}{2}$.
   This step requires computing P solutions to a coarse PDE in order to get the energy $\mathcal{E}(\overline{A}^n, \cdot)$.
   We next solve a $P \times P$ eigenvalue problem.

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   In practice, we perform a gradient descent together with a line search.
   The gradient can be expressed with solutions computed in previous step, hence no additionnal costs.

Choice of the loadings

We identify P appropriate loadings $(g_i)_{1 \leqslant i \leqslant P}$ such that

$$\sup_{g \in L^2_0(\partial\Omega)} \left| \mathcal{E}(A_\varepsilon, g) - \mathcal{E}(\overline{A}, g) \right| \approx \sup_{g \in \underset{1 \leqslant i \leqslant P}{\mathrm{Span}} (g_i)} \left| \mathcal{E}(A_\varepsilon, g) - \mathcal{E}(\overline{A}, g) \right|.$$

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Rayleigh quotient: we optimize

$$\sup_{\|g\|_{L^2(\partial\Omega)}=1} \left|\mathcal{E}(A_\varepsilon,g)-\mathcal{E}(\overline{A},g)\right| = \sup_{g\in L^2_0(\partial\Omega)} \left|\frac{\int_{\partial\Omega} g\,(\mathcal{T}_\varepsilon-\mathcal{T}_{\overline{A}})\,g}{\int_{\partial\Omega} g^2}\right|,$$

where

$$\begin{aligned}
\mathcal{T}_\varepsilon &: g\in L^2_0(\partial\Omega)\longrightarrow u_\varepsilon(g)|_{\partial\Omega} && \text{with } u_\varepsilon(g) \text{ sol. to (1)},\\
\mathcal{T}_{\overline{A}} &: g\in L^2_0(\partial\Omega)\longrightarrow u(\overline{A},g)|_{\partial\Omega} && \text{with } u(\overline{A},g) \text{ sol. to (2)}.
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Thus, we seek the eigenmode of $\mathcal{T}_\varepsilon-\mathcal{T}_{\overline{A}}$ with largest eigenvalue in absolute value.

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Thus, we seek the eigenmode of $\mathcal{T}_\varepsilon - \mathcal{T}_{\overline{A}}$ with largest eigenvalue in absolute value.

Case of spheric, periodic coefficients: it holds that

$$\mathcal{T}_\varepsilon - \mathcal{T}_{\overline{A}} \underset{\varepsilon\to 0}{\longrightarrow} (A_\star - \overline{A})\,\mathcal{T}$$

with $\mathcal{T} : g \in L^2_0(\partial\Omega) \longrightarrow w(g)|_{\partial\Omega}$ where $w(g)$ is solution to

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Practical choice: We select the $P \gtrapprox P_d = \frac{d(d+1)}{2}$ first eigenmodes of $\mathcal{T}$.

Numerical results (periodic)

In 2D ($\Omega =]0,1[^2$), we consider the coefficient

$$A_\varepsilon(x,y) = A^{\mathrm{per}}\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) = \begin{pmatrix} 22 + 10 \times (\sin(2\pi \frac{x}{\varepsilon}) + \sin(2\pi \frac{y}{\varepsilon})) & 0 \\ 0 & 12 + 2 \times (\sin(2\pi \frac{x}{\varepsilon}) + \sin(2\pi \frac{y}{\varepsilon})) \end{pmatrix},$$

for which

$$A_\star \approx \begin{pmatrix} 19.3378 & 0 \\ 0 & 11.8312 \end{pmatrix}.$$

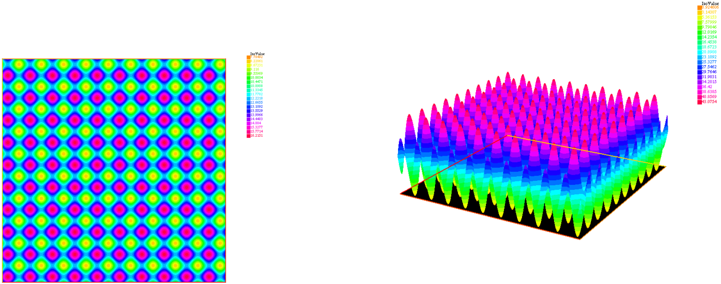


Figure: Components 11 and 22 of coefficient A_ε.

Numerical results (periodic)

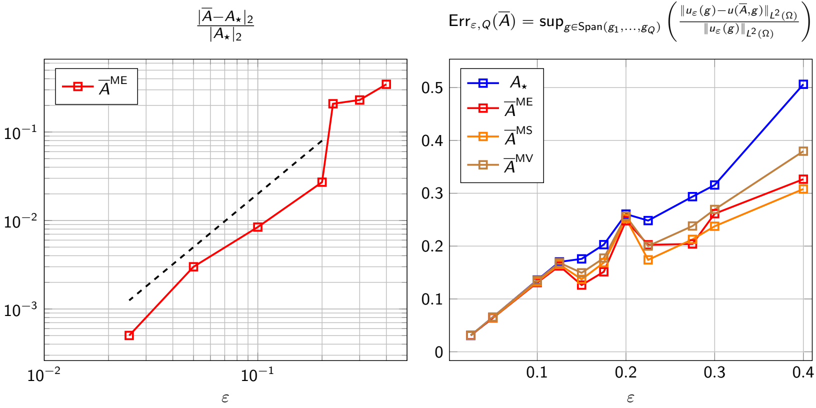


Figure: (left) Error between coefficients $A_\star$ and $\overline{A}^{\mathrm{ME}}_{\varepsilon,P}$.
(right) Criterion $\mathrm{Err}_{\varepsilon,Q}(\overline{A})$ for $\overline{A} \in \{A_\star, \overline{A}^{\mathrm{MV}}_{\varepsilon,P}, \overline{A}^{\mathrm{ME}}_{\varepsilon,P}, , \overline{A}^{\mathrm{MS}}_{\varepsilon,P}\}$ (with $Q = 11$).

We now use a non periodic coefficient (random checkerboard),

$$A_\varepsilon(x, y, \omega) = a^{\mathrm{sto}}\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}, \omega\right) = \left(\sum_{k\in\mathbb{Z}^2} X_k(\omega)\mathbb{1}_{k+Q}\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)\right)\mathrm{Id},$$

with X_k i.i.d random variables such that $\mathbb{P}(X_k = \gamma_1) = \mathbb{P}(X_k = \gamma_2) = \frac{1}{2}$ and $(\gamma_1, \gamma_2) = (4, 16)$.
We have

$$A_\star = \sqrt{\gamma_1\gamma_2}\ \mathrm{Id}.$$

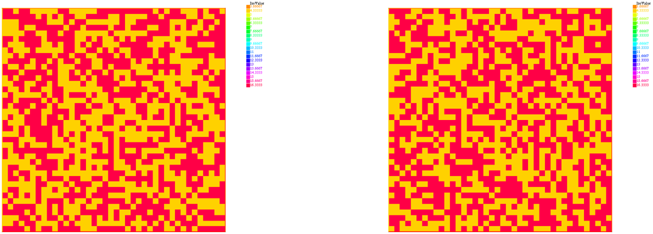
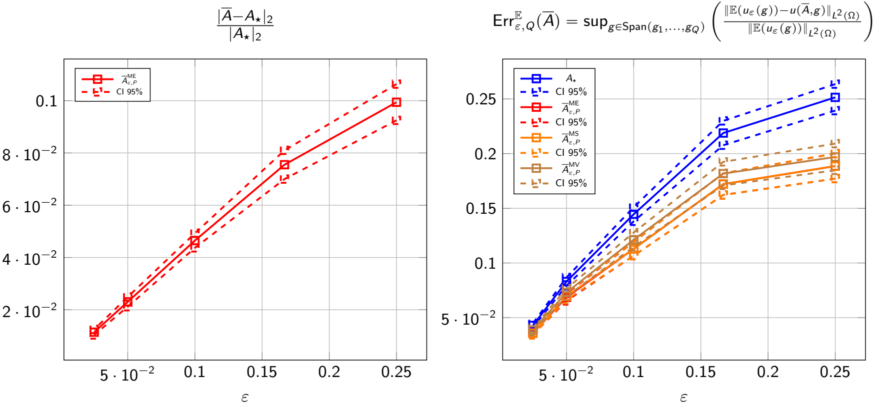


Figure: Two realizations of coefficient A_ε.

Our strategy rewrites $I_\varepsilon = \inf\sup|\mathbb{E}(\mathcal{E}(A_\varepsilon(\cdot, \omega), f)) - \mathcal{E}(\overline{A}, f)|$. Confidence intervals are computed from 40 realizations of the expectation (itself approximated by its empirical mean using 40 realizations of the coefficient a^{sto}).

Numerical results (stochastic)



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Noise

Motivation: The value of the energy may not be known exactly.

Formulation: Consider a multiplicative noise in the energy:

$$\mathcal{E}(A_\varepsilon, g; \sigma) = (1 + \sigma\eta)\ \mathcal{E}(A_\varepsilon, g).$$

where A_ε is a deterministic periodic coefficient, and η follows a Gaussian distribution.

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Results:

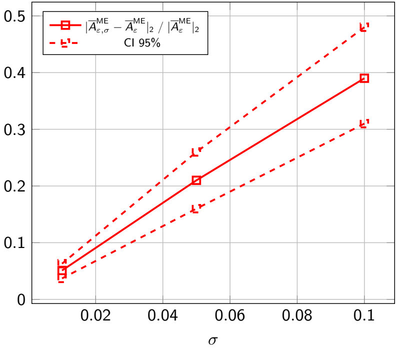

Figure: Error $\frac{|\overline{A}^{\mathrm{ME}}_{\varepsilon,\sigma} - \overline{A}^{\mathrm{ME}}_{\varepsilon}|_2}{|\overline{A}^{\mathrm{ME}}_{\varepsilon}|_2}$ as a function of the noise magnitude σ (for $\varepsilon = 0.025$).

Part II

Perturbative reconstruction of effective coefficients

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- **Assumption**: The effective coefficient lies in neighboorhood of a known coefficient $\overline{A}_0$.

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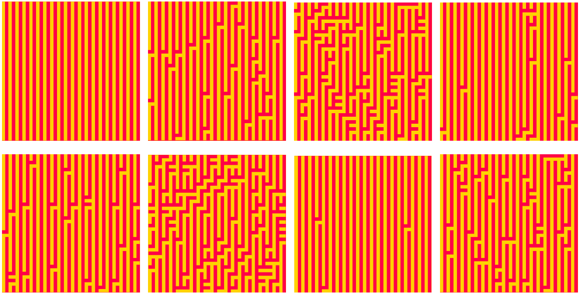
- **Example**: Periodic material with random defect

$$A_{\varepsilon,\eta}(x,\omega) = A_\varepsilon^{\text{per}}(x) + b_\eta(\omega) C_\varepsilon^{\text{per}}(x),$$

with $C_\varepsilon^{\text{per}}$ possibly not negligible, but

$$A_{\star,\eta} = \overline{A}_0 + \eta \overline{A}_1 + o(\eta),$$

where $\overline{A}_0$ is known (e.g. given as an industrial reference).



Perturbative reconstruction of effective coefficients

- **Assumption**: The effective coefficient lies in neighboorhood of a known coefficient $\overline{A}_0$.

- **Issue**: Computing the effective coefficient using previous methods for many realizations ω and different defect rates η may lead to prohibitive computational costs...

Question

How can we use the a priori knowledge of $\overline{A}_0$ to guide and speed up the optimization ?

Perturbative development

Consider the problem

$$-\operatorname{div}\left(A_\varepsilon \nabla u_\varepsilon\right) = f \text{ in } \Omega, \qquad \text{and} \qquad u_\varepsilon = 0 \text{ on } \partial\Omega,$$

and its approximation by

$$-\operatorname{div}\left(\overline{A} \nabla \overline{u}\right) = f \text{ in } \Omega, \qquad \text{and} \qquad \overline{u} = 0 \text{ on } \partial\Omega.$$

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$\overline{A}$	$\overline{A}_0 + \eta \overline{B}$
$u(\overline{A}, f)$	$u_0 + \eta v$

where $u_0 = u(\overline{A}_0, f)$ and $v = v(\overline{A}_0, \overline{B}, f)$ is solution to

$$\begin{cases} -\mathrm{div}\left(\overline{A}_0 \nabla v\right) = \mathrm{div}\left(\overline{B}\nabla u_0\right) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

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Linearity implies that $v = \sum_{ij} \overline{B}_{ij} v_{ij}$ with $v_{ij} = v_{ij}(\overline{A}_0, f)$ solution to

$$\begin{cases} -\mathrm{div}\left(\overline{A}_0 \nabla v_{ij}\right) = \mathrm{div}\left(E_{ij} \nabla u_0\right) & \text{in } \Omega, \\ v_{ij} = 0 & \text{on } \partial\Omega. \end{cases}$$

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$\mathcal{E}(\overline{A}, f)$	$\mathcal{E}(\overline{A}_0, f) + \eta \sum_{ij} \overline{B}_{ij} \mathcal{F}_{ij}(\overline{A}_0, f)$

where $\mathcal{E}(\overline{A}_0, f) = -\frac{1}{2}\int_\Omega f \; u_0$ and

$$\begin{aligned} \mathcal{F}_{ij}(\overline{A}_0, f) &= -\frac{1}{2}\int_\Omega f \; v_{ij} \\ &= \frac{1}{2}\int_\Omega \partial_i u_0 \; \partial_j u_0. \end{aligned}$$

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We formulate the optimization problem

$$\inf_{\substack{\overline{B} \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \\ \alpha \leqslant \overline{A}_0 + \overline{B} \leqslant \beta}} \ \sup_{\|f\|_{L^2(\Omega)}=1.} \left(\mathcal{E}(A_{\varepsilon,\eta}, f) - \mathcal{E}(\overline{A}_0, f) - \sum_{1\leqslant i,j\leqslant d} [\overline{B}]_{ij} \mathcal{F}_{ij}(\overline{A}_0, f) \right)^2.$$

Implementation aspects

- **Offline stage**:
  - Compute $u(\overline{A}_0, f)$.
  - Compute $\mathcal{E}(\overline{A}_0, f)$ and $\mathcal{F}_{ij}(\overline{A}_0, f)$ for any $1 \leqslant i \leqslant j \leqslant d$.

↳ computing $P \approx \frac{d(d+1)}{2}$ solutions to a coarse PDE and $P(1 + \frac{d(d+1)}{2})$ integrals.

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↳ computing $P \approx \frac{d(d+1)}{2}$ solutions to a coarse PDE and $P(1 + \frac{d(d+1)}{2})$ integrals.

- **Online stage**: we apply a gradient descent
    - Define f^n, the argsup to

$$\sup_{f \quad \text{s.t. } \|f\|_{L^2(\Omega)} = 1} \left(\mathcal{E}(A_\varepsilon, f) - \mathcal{E}(\overline{A}_0, f) - \sum_{ij} [\overline{B}^n]_{ij} \; \mathcal{F}_{ij}(\overline{A}_0, f) \right)^2.$$

- Define

$$\overline{B}^{n+1} = \overline{B}^n - \mu \nabla_{\overline{B}} J_\varepsilon^n(\overline{B}^n)$$

with

$$J_\varepsilon^n(\overline{B}) = \left(\mathcal{E}(A_\varepsilon, f^n) - \mathcal{E}(\overline{A}_0, f^n) - \sum_{ij} [\overline{B}]_{ij} \; \mathcal{F}_{ij}(\overline{A}_0, f^n) \right)^2.$$

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  - Compute $u(\overline{A}_0, f)$.
  - Compute $\mathcal{E}(\overline{A}_0, f)$ and $\mathcal{F}_{ij}(\overline{A}_0, f)$ for any $1 \leqslant i \leqslant j \leqslant d$.

↳ computing $P \approx \frac{d(d+1)}{2}$ solutions to a coarse PDE and $P(1 + \frac{d(d+1)}{2})$ integrals.

- **Online stage**: we apply a gradient descent
  - Define f^n, the argsup to

$$\sup_{f \ \text{ s.t. } \|f\|_{L^2(\Omega)} = 1} \left(\mathcal{E}(A_\varepsilon, f) - \mathcal{E}(\overline{A}_0, f) - \sum_{ij} [\overline{B}^n]_{ij} \ \mathcal{F}_{ij}(\overline{A}_0, f) \right)^2 .$$

  In practice, $\sup_{f \in L^2(\Omega)} \to \sup_{f \in V_P}$ on $V_P = \text{Span}\{P \text{ loadings}\}$, with $P \approx \frac{d(d+1)}{2}$. This step amounts to solving a $P \times P$ eigenvalue problem.
  - Define

$$\overline{B}^{n+1} = \overline{B}^n - \mu \nabla_{\overline{B}} J_\varepsilon^n(\overline{B}^n)$$

  with

$$J_\varepsilon^n(\overline{B}) = \left(\mathcal{E}(A_\varepsilon, f^n) - \mathcal{E}(\overline{A}_0, f^n) - \sum_{ij} [\overline{B}]_{ij} \ \mathcal{F}_{ij}(\overline{A}_0, f^n) \right)^2 .$$

  Observe that J_ε^n is quadratic.

- **Offline stage**:
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  - Compute $\mathcal{E}(\overline{A}_0, f)$ and $\mathcal{F}_{ij}(\overline{A}_0, f)$ for any $1 \leqslant i \leqslant j \leqslant d$.

↳ computing $P \approx \frac{d(d+1)}{2}$ solutions to a coarse PDE and $P(1 + \frac{d(d+1)}{2})$ integrals.

- **Online stage**: we apply a gradient descent
  - Define f^n, the argsup to

$$\sup_{f \ \text{ s.t. } \|f\|_{L^2(\Omega)} = 1} \left(\mathcal{E}(A_\varepsilon, f) - \mathcal{E}(\overline{A}_0, f) - \sum_{ij} [\overline{B}^n]_{ij} \ \mathcal{F}_{ij}(\overline{A}_0, f) \right)^2 .$$

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Observe that J_ε^n is quadratic.

↳ no additional computations of coarse PDE !

- Preserve the quality of the approximation.
- Reduction of computational costs (by a factor of ≈ 80 to 400).

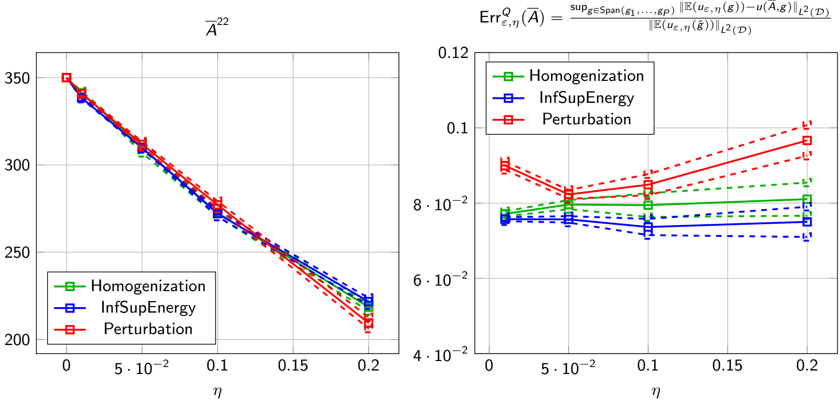

Figure: (left) Component 22 for various approximations of the effective coefficient. (right) Criterion $Err^{\mathbb{E}}_{\varepsilon,Q}(\overline{A})$ for different constant coefficients (with $Q = 9$ and $\varepsilon = 0.025$).

Part III

Efficient selection of effective coefficients

Framework

- **Setting**: we are given
  - a list of candidate coefficients $\mathcal{A} = \{\overline{A}_1, ..., \overline{A}_N\}$.
  - a list of admissible loadings $\mathcal{F} = \{f_1, ..., f_P\}$.
  - a measurement operator $\mathcal{O} : \mathcal{A} \times \mathcal{F} \to \mathbb{R}$ or $L^2(\Omega)$ (e.g. $\mathcal{O}(A_\varepsilon, f) = u_\varepsilon(f)$ or $\mathcal{E}(A_\varepsilon, f)$).

Work inspired by discussions with H. Ammari (ETH Zürich).

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- **Challenge**:
  - Expensive measurement costs $\implies$ budget of $Q \ll P$ measurements.
  - (Unknown) decomposition of $\mathcal{F}$ into $\mathcal{F}_{\text{disc}}$ and $\mathcal{F}_{\text{non-disc}}$ such that

$$\text{card}\left(\mathcal{F}_{\text{disc}}\right) \ll \text{card}\left(\mathcal{F}\right),$$

  and for any $f \in \mathcal{F}_{\text{non-disc}}$ and any $\overline{A}, \overline{B} \in \mathcal{A}^2$,

$$\|\mathcal{O}(A_\varepsilon, f) - \mathcal{O}(\overline{A}, f)\|_{\mathcal{O}} \approx \|\mathcal{O}(A_\varepsilon, f) - \mathcal{O}(\overline{B}, f)\|_{\mathcal{O}}.$$

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Objective

Select the coefficient in $\mathcal{A}$ that provides the best effective approximation of the underlying system while simultaneously minimizing the number of measurement operations $\mathcal{O}(A_\varepsilon, f)$ with $f \in \mathcal{F}$.

Work inspired by discussions with H. Ammari (ETH Zürich).

Iterative algorithm: each step k selects a loading f^k in $\mathcal{F}$ and update the choice of the best coefficient $\overline{A}^k$ in $\mathcal{A}$.

Selection algorithm

Iterate *k:*

1. *Compute discrimination rate $\Delta^k(f)$ for any f in $\mathcal{F}^k = \mathcal{F} \backslash \{f^p\}_{p=1,\dots,k-1}$.*
2. *Select*
$$f^k \in \arg\max_{f \in \mathcal{F}^k} \Delta^k(f).$$
3. *Measure $\mathcal{O}(A_\varepsilon, f^k)$.*
4. *Define the effective coefficient*
$$\overline{A}^k \in \arg\min_{\overline{A} \in \mathcal{A}} \gamma^k(\overline{A}).$$

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Core components:

- Discrimination rate: Δ^k estimates how a loadings is discriminative w.r.t $\mathcal{O}$ and $\mathcal{A}$.
- Effectiveness score : γ^k assesses the quality of a coefficient as an effective coefficient for the system.

Conclusion and perspectives

Conclusion and perspectives

Our strategies

- aim at determining effective approximations for multiscale PDEs through effective coefficients,
- are designed for context where only limited information is available,
- are inspired by homogenization theory and consistent with it (numerically and theoretically),
- can be extended outside the regime of separated scale.

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Perspectives

- Application to real experimental data.
- Extension to non-constant effective coefficients.
- Convergence analysis of $\overline{A}$ to $A_\star$.
- Convergence analysis of selection algorithm.

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Thank you !

A different perspective on noise

Motivation: anticipate on reproducibility errors during model deployment.

Idea: treat $\overline{A}$ as a random field and optimize upon its mean.

Formulation: consider the problem

$$\inf_{\overline{A} \in \mathcal{S}_{\alpha,\beta}} \sup_{\|g\|_{L^2(\partial\Omega)} = 1} \left| \mathcal{E}(A_\varepsilon, g) - \mathbb{E}\left(\mathcal{E}(\overline{A} + \sigma\eta, g)\right) \right|^2,$$

where η is a Gaussian variable.

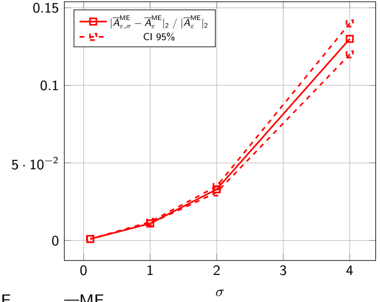

Figure: Error between $\overline{A}^{\mathrm{ME}}_{\varepsilon,\sigma}$ and $\overline{A}^{\mathrm{ME}}_{\varepsilon}$ as a function of the noise magnitude σ (for $\varepsilon = 0.05$).

Convergence analysis

We consider the slightly modified

$$\inf_{\substack{\overline{A} \in \mathbb{R}^{d\times d}_{\mathrm{sym}},\\ \alpha \leqslant \overline{A} \leqslant \beta}} \sum_{1\leqslant i\leqslant j\leqslant d} \left|\mathcal{E}(A_\varepsilon, g_{i,j}) - \mathcal{E}(\overline{A}, g_{i,j})\right|,$$

where $(g_{i,j})_{1\leqslant i\leqslant j\leqslant d}$ are preselected loadings.

For an appropriate choice of loadings $(g_{i,j})_{1\leqslant i\leqslant j\leqslant d}$, it holds that

$$\left|\overline{A}^{\mathrm{opt}}_\varepsilon - A_\star\right| \leqslant C\delta(\varepsilon),$$

where $\delta(\varepsilon)$ is any function such that

$$|\mathcal{E}(A_\varepsilon) - \mathcal{E}(A_\star)| \leqslant C\delta(\varepsilon).$$

Loading selection

We define

$$g_{i,j} = \left(\frac{e_i + e_j}{2}\right) \cdot n,$$

where $(e_i)_{1 \leqslant i \leqslant d}$ is the canonical basis of $\mathbb{R}^d$.

For any $\overline{A} \in \mathbb{R}^{d \times d}_{\text{sym}}$, the solution to

$$-\text{div}\left(\overline{A}\nabla\overline{u}\right) = 0 \text{ in } \Omega, \qquad \left(\overline{A}\nabla\overline{u}\right) \cdot n = g_{i,j} \text{ on } \partial\Omega.$$

writes

$$\overline{u}_{i,j} = \left(\overline{A}^{-1} e_{i,j}\right) \cdot x.$$

Thus the energy writes:

$$\mathcal{E}(\overline{A}, g_{i,j}) = \int_\Omega \underbrace{\left(\overline{A}^{-1} e_{i,j}\right)^T}_{(\nabla u_{i,j})^T} \overline{A} \underbrace{\left(\overline{A}^{-1} e_{i,j}\right)}_{\nabla u_{i,j}} = |\Omega| e_{i,j}^T \overline{A}^{-1} e_{i,j}$$

Then, we get

$$\begin{aligned} \|\overline{A} - A_\star\| &\leqslant C\|\overline{A}^{-1} - A_\star^{-1}\| \\ &\leqslant C \sum_{1 \leqslant i \leqslant j \leqslant d} |e_{ij}^T(\overline{A}^{-1} - A_\star^{-1}) e_{ij}| \\ &\leqslant \ldots \\ &\leqslant C\delta(\varepsilon). \end{aligned}$$

Efficient selection of effective coefficients

Framework

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Objective

Select the coefficient in $\mathcal{A}$ that provides the best effective approximation of the underlying system while simultaneously minimizing the number of measurement operations $\mathcal{O}(A_\varepsilon, f)$ with $f \in \mathcal{F}$.

Strategy

Iterative algorithm: each step k selects a loading f^k in $\mathcal{F}$ and update the choice of the best coefficient $\overline{A}^k$ in $\mathcal{A}$.

Core components:

- Discrimination rate: it estimates how a loadings is discriminative w.r.t observable $\mathcal{O}$.
- Effectiveness score: it assesses the quality of a coefficient as an effective coefficient for the system.

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$\Delta_1(f)$	$\max\limits_{(\overline{A},\overline{B})\in\mathcal{A}^2} \mathrm{Err}(\overline{A},\overline{B},f)$	$\mathrm{Err}(\overline{A},\overline{B},f) = \frac{\\|\mathcal{O}(\overline{A},f)-\mathcal{O}(\overline{B},f)\\|_{\mathcal{O}}}{\\|\mathcal{O}(\overline{A},f)\\|_{\mathcal{O}}}$

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$\Delta_\varepsilon(f)$	$\max\limits_{(\overline{A},\overline{B})\in\mathcal{A}^2} \left\lvert \mathrm{Err}_\varepsilon(\overline{A},f)-\mathrm{Err}_\varepsilon(\overline{B},f)\right\rvert$	$\mathrm{Err}_\varepsilon(\overline{A},f) = \frac{\lVert \mathcal{O}(A_\varepsilon,f)-\mathcal{O}(\overline{A},f)\rVert_{\mathcal{O}}}{\lVert \mathcal{O}(A_\varepsilon,f)\rVert_{\mathcal{O}}}$

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$\Delta_3(f)$	$\max\limits_{(\overline{A},\overline{B})\in\mathcal{A}^2} \left\lvert \mathrm{Err}_\varepsilon(\overline{A},f) - \mathrm{Err}_\varepsilon(\overline{B},f)\right\rvert$	$\mathrm{Err}_\varepsilon(\overline{A},f) = \frac{\\|\mathcal{O}(A_\varepsilon,f)-\mathcal{O}(\overline{A},f)\\|_{\mathcal{O}}}{\\|\mathcal{O}(A_\varepsilon,f)\\|_{\mathcal{O}}}$

- Effectiveness score: it assesses the quality of a coefficient as an effective coefficient for the system.
  ↳ e.g.

$$\gamma^k(\overline{A}) = \max_{f\in\{f_1,\dots,f_k\}} \mathrm{Err}_\varepsilon(\overline{A},f).$$

Selection algorithm

Initialization:

Select $f^1 \in \mathcal{F}$ by solving

$$f^1 = \arg\max_{f \in \mathcal{F}} \max_{(\overline{A},\overline{B}) \in \mathcal{A}^2} \frac{\|\mathcal{O}(\overline{A}, f) - \mathcal{O}(\overline{B}, f)\|_{\mathcal{O}}}{\|\mathcal{O}(\overline{A}, f)\|_{\mathcal{O}}}.$$

Iterate k:

1. *Compute discrimination rate* $\Delta^k(f)$ *for any* f *in* $\mathcal{F}^k = \mathcal{F} \backslash \{f^p\}_{p=1,\dots,k-1}$.
2. *Select*
$$f^k \in \arg\max_{f \in \mathcal{F}^k} \Delta^k(f).$$
3. *Measure* $\mathcal{O}(A_\varepsilon, f^k)$.
4. *Define the effective coefficient*
$$\overline{A}^k \in \arg\min_{\overline{A} \in \mathcal{A}} \gamma^k(\overline{A}).$$

- **Microstructure**. Consider

$$A_\varepsilon(x) = \begin{cases} \gamma_1 + \gamma_2 \sin\left(\dfrac{2\pi x}{\varepsilon}\right) & \text{if } x \in D_1, \\ \gamma_3 & \text{if } x \in D_2. \end{cases}$$

with

$$\gamma_3 = a_\star$$

the limit in the sense of homogenization of $x \mapsto \gamma_1 + \gamma_2 \sin(\frac{2\pi x}{\varepsilon})$.

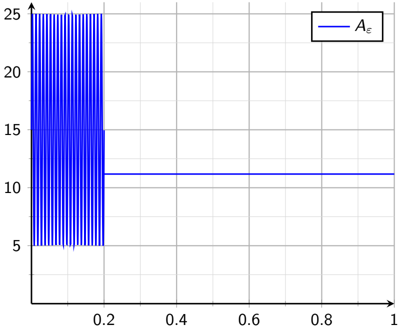


Figure: Coefficient A_ε with $\varepsilon = 0.01$.

- **Microstructure**.
- **Set** $\mathcal{F}$. Consider

$$f_n = \mathbb{1}_{\left(\frac{n-1}{N}, \frac{n}{N}\right)}$$

and

$$\mathcal{F} = \begin{matrix} \overbrace{\{f_n \text{ s.t. } \mathrm{Supp}(f_n) \subset D_1\}}^{\mathcal{F}_{\text{disc}}} \\ \cup \\ \underbrace{\{f_n \text{ s.t. } \mathrm{Supp}(f_n) \subset D_2\}}_{\mathcal{F}_{\text{non-disc}}} \end{matrix}.$$

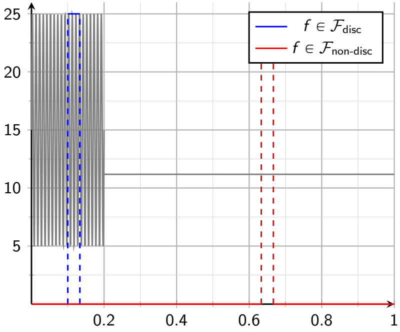


Figure: Admissible loadings.

Numerical results

Microstructure.
Set $\mathcal{F}$.
Set $\mathcal{A}$. Consider

$$\overline{A}(x) = \begin{cases} \overline{A}^1 \text{ if } x < 0.2, \\ \overline{A}^2 \text{ if } x > 0.2. \end{cases}$$

with $\overline{A}^1$ and $\overline{A}^2$ are constants.

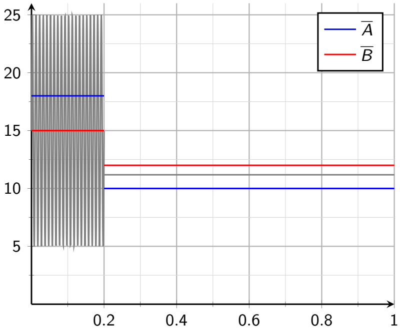


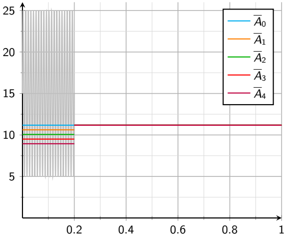
Figure: Effective coefficients.

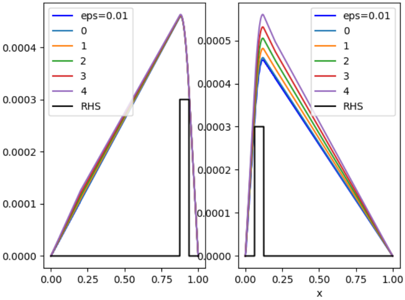
Setting:

$$\mathcal{A}_1 = \{\overline{A}_0 = (a_\star, \quad a_\star), \\ \overline{A}_1 = (0.95a_\star, \quad a_\star), \\ \overline{A}_2 = (0.9a_\star, \quad a_\star), \\ \overline{A}_3 = (0.85a_\star, \quad a_\star), \\ \overline{A}_4 = (0.8a_\star, \quad a_\star)\},$$

and

$$\frac{\mathrm{card}\,(\mathcal{F}_{\mathrm{disc}})}{\mathrm{card}\,(\mathcal{F})} \approx 0.2.$$





Setting:

$$\begin{aligned}\mathcal{A}_1 = \{\overline{A}_0 &= (a_\star, \quad a_\star),\\ \overline{A}_1 &= (0.95a_\star, \quad a_\star),\\ \overline{A}_2 &= (0.9a_\star, \quad a_\star),\\ \overline{A}_3 &= (0.85a_\star, \quad a_\star),\\ \overline{A}_4 &= (0.8a_\star, \quad a_\star)\},\end{aligned}$$

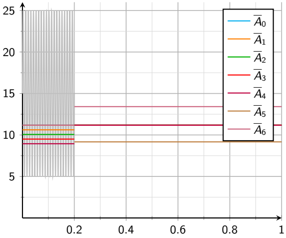
and

$$\frac{\mathrm{card}\left(\mathcal{F}_{\mathrm{disc}}\right)}{\mathrm{card}\left(\mathcal{F}\right)} \approx 0.2.$$

Conclusions:

- Δ_1, Δ_2 and Δ_ε perform the same selection.
- Loadings in $\mathcal{F}_{\mathrm{disc}}$ are first identified.
- $\overline{A}_0 = (a_\star, a_\star)$ is the best coefficient at each step.

Step	Loadings	Best Coefficient
1	$f_1 \in \mathcal{F}_{\mathrm{disc}}$	$\overline{A}_0$
2	$f_2 \in \mathcal{F}_{\mathrm{disc}}$	$\overline{A}_0$
3	$f_3 \in \mathcal{F}_{\mathrm{disc}}$	$\overline{A}_0$
4	$f_4 \in \mathcal{F}_{\mathrm{disc}}$	$\overline{A}_0$
5	$f_{12} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
6	$f_{13} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
7	$f_{14} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
8	$f_{15} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
9	$f_{16} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
10	$f_{17} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
11	$f_{18} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
12	$f_{19} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
13	$f_{20} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
14	$f_{21} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$
15	$f_{22} \in \mathcal{F}_{\mathrm{non\text{-}disc}}$	$\overline{A}_0$

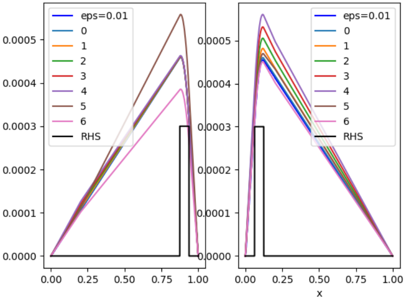


Setting:

$$\begin{aligned}\mathcal{A}_2 = \mathcal{A}_1 \cup \{\overline{A}_5 &= (a_\star, \quad 0.82 a_\star),\\ \overline{A}_6 &= (a_\star, \quad 1.2 a_\star)\},\end{aligned}$$

and

$$\frac{\operatorname{card}(\mathcal{F}_{\text{disc}})}{\operatorname{card}(\mathcal{F})} \approx 0.2.$$



Numerical results: case 2

Setting:

$$\mathcal{A}_2 = \mathcal{A}_1 \cup \{\overline{A}_5 = (a_\star, \quad 0.82 a_\star), \\ \overline{A}_6 = (a_\star, \quad 1.2 a_\star)\},$$

and

$$\frac{\mathrm{card}\,(\mathcal{F}_{\mathrm{disc}})}{\mathrm{card}\,(\mathcal{F})} \approx 0.2.$$

Conclusions:

- Δ_2 better replicates Δ_ε than Δ_1.
- $\overline{A}_0 = (a_\star, a_\star)$ is the best coefficient at each step.
- Δ_1 selects loadings that do not precisely discriminate $\overline{A}_0$ from other coefficients in $\mathcal{A}_1$.

Step	Δ_ε	Δ_2
1	$f_1 \in \mathcal{F}_{\text{disc}}$	$f_{29} \in \mathcal{F}_{\text{non-disc}}$
2	$f_2 \in \mathcal{F}_{\text{disc}}$	$f_1 \in \mathcal{F}_{\text{disc}}$
3	$f_3 \in \mathcal{F}_{\text{disc}}$	$f_2 \in \mathcal{F}_{\text{disc}}$
4	$f_4 \in \mathcal{F}_{\text{disc}}$	$f_3 \in \mathcal{F}_{\text{disc}}$
5	$f_{29} \in \mathcal{F}_{\text{non-disc}}$	$f_{28} \in \mathcal{F}_{\text{non-disc}}$
6	$f_{28} \in \mathcal{F}_{\text{non-disc}}$	$f_4 \in \mathcal{F}_{\text{disc}}$
7	$f_{12} \in \mathcal{F}_{\text{non-disc}}$	$f_{27} \in \mathcal{F}_{\text{non-disc}}$
8	$f_{27} \in \mathcal{F}_{\text{non-disc}}$	$f_{12} \in \mathcal{F}_{\text{non-disc}}$
9	$f_{26} \in \mathcal{F}_{\text{non-disc}}$	-
10	$f_{25} \in \mathcal{F}_{\text{non-disc}}$	-
11	$f_{24} \in \mathcal{F}_{\text{non-disc}}$	-
12	$f_{23} \in \mathcal{F}_{\text{non-disc}}$	-
13	$f_{22} \in \mathcal{F}_{\text{non-disc}}$	-
14	$f_{21} \in \mathcal{F}_{\text{non-disc}}$	-
15	$f_{20} \in \mathcal{F}_{\text{non-disc}}$	-
16	$f_{19} \in \mathcal{F}_{\text{non-disc}}$	-
17	$f_{18} \in \mathcal{F}_{\text{non-disc}}$	-
17	$f_{17} \in \mathcal{F}_{\text{non-disc}}$	-

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3	$f_3 \in \mathcal{F}_{\text{disc}}$	$f_{27} \in \mathcal{F}_{\text{non-disc}}$
4	$f_4 \in \mathcal{F}_{\text{disc}}$	$f_{26} \in \mathcal{F}_{\text{non-disc}}$
5	$f_{29} \in \mathcal{F}_{\text{non-disc}}$	$f_{25} \in \mathcal{F}_{\text{non-disc}}$
6	$f_{28} \in \mathcal{F}_{\text{non-disc}}$	$f_{24} \in \mathcal{F}_{\text{non-disc}}$
7	$f_{12} \in \mathcal{F}_{\text{non-disc}}$	$f_{23} \in \mathcal{F}_{\text{non-disc}}$
8	$f_{27} \in \mathcal{F}_{\text{non-disc}}$	$f_{22} \in \mathcal{F}_{\text{non-disc}}$
9	$f_{26} \in \mathcal{F}_{\text{non-disc}}$	$f_{21} \in \mathcal{F}_{\text{non-disc}}$
10	$f_{25} \in \mathcal{F}_{\text{non-disc}}$	$f_{20} \in \mathcal{F}_{\text{non-disc}}$
11	$f_{24} \in \mathcal{F}_{\text{non-disc}}$	$f_{19} \in \mathcal{F}_{\text{non-disc}}$
12	$f_{23} \in \mathcal{F}_{\text{non-disc}}$	$f_{18} \in \mathcal{F}_{\text{non-disc}}$
13	$f_{22} \in \mathcal{F}_{\text{non-disc}}$	$f_{17} \in \mathcal{F}_{\text{non-disc}}$
14	$f_{21} \in \mathcal{F}_{\text{non-disc}}$	$f_{16} \in \mathcal{F}_{\text{non-disc}}$
15	$f_{20} \in \mathcal{F}_{\text{non-disc}}$	$f_{15} \in \mathcal{F}_{\text{non-disc}}$
16	$f_{19} \in \mathcal{F}_{\text{non-disc}}$	$f_{14} \in \mathcal{F}_{\text{non-disc}}$
17	$f_{18} \in \mathcal{F}_{\text{non-disc}}$	$f_{13} \in \mathcal{F}_{\text{non-disc}}$
17	$f_{17} \in \mathcal{F}_{\text{non-disc}}$	$f_1 \in \mathcal{F}_{\text{non-disc}}$

Schrödinger equation

Homogenization. Consider the Schrödinger equation

$$-\Delta u_\varepsilon + V_\varepsilon u_\varepsilon = f \text{ in } \Omega, \qquad u_\varepsilon = 0 \text{ on } \partial\Omega.$$

In the periodic case (i.e. $V_\varepsilon(x) = \frac{1}{\varepsilon} V_{\text{per}}\left(\frac{x}{\varepsilon}\right)$), we define

$$-\Delta u_\star + V_\star u_\star = f \text{ in } \Omega, \qquad u_\star = 0 \text{ on } \partial\Omega,$$

with $V_\star \in \mathbb{R}$ defined through a corrector w, periodic solution to

$$-\Delta w = V_{\text{per}} \text{ in } \mathbb{R}^d.$$

Homogenization assesses that

$$u_\varepsilon - u_\star \to 0 \text{ in } L^2(\Omega),$$
$$u_\varepsilon - \underbrace{\left(1 + \varepsilon w\left(\frac{x}{\varepsilon}\right)\right) u_\star}_{u_{\varepsilon,1}} \to 0 \text{ in } H^1(\Omega).$$

Effective approximation in $H^1(\Omega)$. Based on measurements of solutions $(u_\varepsilon(f_p))_{1\leqslant p\leqslant P}$ and their gradients, we proceed in two steps:

1. a *best* potential $\overline{V}$ is defined through an optimization problem.
2. a *corrector term* is built using measurements of solution $(u_\varepsilon(f_p))_{1\leqslant p\leqslant P}$.

Defining a best potential $\overline{V}$

We consider the optimization problem

$$\inf_{\overline{V}\in\mathbb{R}} \sup_{f\in L^2(\Omega)} \left\| (-\Delta)^{-1}(-\Delta+\overline{V})(u_\varepsilon(f)-u(\overline{V},f)) \right\|_{L^2(\Omega)}^2 ,$$

with $\overline{u}=u(\overline{V},f)$ solution to

$$-\Delta\overline{u}+\overline{V}\overline{u}=f \text{ in } \Omega, \qquad \overline{u}=0 \text{ on } \partial\Omega.$$

It holds that

Proposition (Existence and uniqueness)

In the periodic setting, there exists a unique minimizer $\overline{V}_\varepsilon^{opt}$ for sufficiently small ε.

Proposition (Asymptotic consistency)

In the periodic setting, the following convergence holds:

$$\lim_{\varepsilon\to 0} \overline{V}_\varepsilon^{opt} = V_\star.$$

By homogenization, we know that

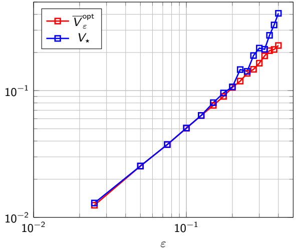
$$\nabla u_\varepsilon \approx \nabla u_\star + u_\star \left(\nabla w\right)\left(\frac{x}{\varepsilon}\right) \text{ in } L^2(\Omega).$$

We define a corrector

$$\inf_{\overline{C}\in\left(L^2(\Omega)\right)^{d\times d}} \sup_{f\in L^2(\Omega)} \left\|\nabla u_\varepsilon(f) - \nabla\overline{u}(f) - \overline{C}\,\overline{u}(f)\right\|^2_{L^2(\Omega)},$$

where $\overline{u}(f) = u(\overline{V}^{\text{opt}}_\varepsilon, f)$.

$\text{Err}_{\varepsilon,Q}(\overline{V}) = \sup_{f\in V_n^Q(\Omega)} \|u_\varepsilon(f) - u(\overline{V},f)\|_{L^2(\Omega)}/\|u_\varepsilon(\hat{f})\|_{L^2(\Omega)}$ $\text{Err}^{\text{corr}}_{\varepsilon,Q}(\overline{V},\overline{C}) = \sup_{f\in V_n^Q(\Omega)} \|\nabla u_\varepsilon(f) - \nabla u(\overline{V},f) - \overline{C}u(\overline{V},f)\|_{L^2(\tilde{\Omega})}/\|\nabla u_\varepsilon(\hat{f})\|_{L^2(\tilde{\Omega})}$



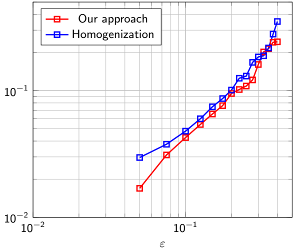


Figure: Comparison of our approach and homogenization in L^2-norm (left) and H^1-norm (right).